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Theory of Energy and Mass Transfer

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FOREWORD FROM THE EDITORS

This book is an important contribution to engineering science, an exemplary search for the new generalizations and principles inherent in actions and interactions of the variables in complex engineering systems. The authors keep foremost a system lying in the fields of traditional chemical and process engineering. Their research is reported in Russian; the senior author, Aleksei Vasilievich Lykov, is a prominent thermophysicist acclaimed in the Soviet Union for his earlier Theory of Drying. It is fitting that this translation rendering their work readily accessible to a wider audience should appear as one of the first in an international series of volumes devoted to the chemical and engineeringphysical sciences.

The text centers round simultaneous heat and mass transfer in porous media, a subject vital to sundry fields of important application. Moreover the approach, the methods, and to some extent the results embodied in it are of sufficient generality to merit the attention of all who confront basic aspects of coupled transfer and rate processes. From the continuum theory of transport phenomena and the thermodynamics of irreversible processes come materials for the development of general equations. Mathematical techniques unsparingly yet economically applied yield solutions of a multitude of special cases, many drawn from the recent and sizable literature in Russian to which the authors and their colleagues have contributed heavily. Not only the convenient arrangement of the solutions but also the commendable efforts of the authors to organize, interpret, and generalize the results all add toward a finished product immediately useful in engineering analysis and design.

All equations, tables, and graphs have been reproduced from the Soviet edition. In Russian usage the decimal point is represented by a comma; the reader should bear in mind this common source of misinterpretation, particularly in using numerical values from the tables and graphs. The reader's indulgence is sought for certain phrasing and punctuation that he may find a bit awkward in English. These arise because the translation, adhering to the mathematical language of the authors, retains the sequence and numbering of equations of the original Russian text.

A phonetic pronunciation guide to the Russian dimensionless numbers is included in the nomenclature section.

The technical review of the translation by Ralph P. Stein, Nuclear Development Corporation of America, is gratefully acknowledged.

L. E. Scriven

Minneapolis, May 1961

FROM THE SOVIET PUBLISHER

This monograph is devoted to the analytic theory of heat and mass transfer phenomena. A system of differential equations for heat and mass transfer is derived using the thermodynamics of irreversible processes as a basis. Methods of finite integral transformations are used to obtain solutions for simple bodies such as plates, cylinders and spheres, with boundary conditions of the second and third kinds. The solutions may be used for calculations of thermal diffusion processes in gaseous mixtures and solutions, and of drying, gasification, combustion, etc.

This book will be found to be of interest by a wide circle of engineers and technical workers and can also serve as a textbook for students of power and thermal engineering at institutions of higher learning.

Preface

The solutions of problems in the transfer of energy, mass and electricity are of considerable importance to the many practical processes which involve such transfer. Such transfer in solid bodies can be usually described by simple linear relationships. For example, heat transfer by conduction can be expressed by the linear Fourier equation: the heat flux is directly proportional to the temperature gradient. The transfer of electricity in metals takes place in accordance with Ohm's law: the flow of electricity is directly proportional to the electrical potential gradient. Molecular mass transfer can be described by the law of diffusion: the flow of mass is directly proportional to the concentration gradient, etc.

These linear relationships have been used as a basis for the derivation of differential equations for the conduction of heat, electricity and diffusion. The solution of these differential equations with given initial and boundary conditions, which describe the initial state of the body and the interaction of the body's surfaces with the surrounding medium, presents certain difficulties. Since the end of the 18th century numerous methods for the solution of differential equations of heat conduction, electricity conduction and diffusion have been developed.

As a result of the development of improved experimental methods it has been established that these transfer phenomena are interrelated. For example, diffusion in a gaseous mixture produces a temperature gradient and heat transfer takes place (the Dufour effect). Conversely, heat transfer by conduction in a gaseous mixture will produce mass transfer (thermal diffusion, or the Soret effect). Similar interrelationships exist in the transfer of heat and electricity. Thus, many times the prime assumptions which are used as the basis for the derivation of the corresponding differential equations, are obtained from an application of the thermodynamics of irreversible processes.

Instead of the separate, independent linear equations for the transfer of heat, electricity and mass, the Onsager system of coupled linear equations is used as the basic assumption for the solution of transfer problems. In the Onsager system, all forms of transfer are determined by a direct effect and other imposed transfer phenomena. The linear Onsager equations lead to a system of reciprocal relationships among the individual differential equations for the transfer of heat, electricity and mass. Consequently, an important problem is to develop methods of solving these systems of differential equations. This appears to be much more complicated in comparison with the problem of developing a mathematical theory of simple diffusion.

This monograph is devoted to the development of an analytical theory of heat and mass transfer, one of the most important aspects of the thermodynamics of irreversible processes.

As far as is known, there are no analogous works in the literature. This monograph was based on the original work of the authors.

The first part of the book deals with the thermodynamics of transfer phenomena and with the derivation of a system of differential equations for simultaneous heat and mass transfer in porous bodies.

The second part deals with the solution of the system of differential equations of heat and mass transfer with various boundary conditions. These solutions may be applied to the selection of optimal regimes of process kinetics. They can also be used to determine the distribution of transfer potentials in bodies of known configuration. The interaction between the potentials (e.g. non-stationary fields of temperature and transfer potentials of bound matter (bound moisture in the case of drying, and similarly in other cases)) is taken into account in the solutions presented in this book. The solutions of the problems are also accompanied by analyses, to clarify their physical significance, and as far as possible, are illustrated by diagrams and tables. The theory of similarity is widely employed and thus most of the solutions are given in dimensionless form. The various relationships and dependencies among the dimensionless numbers are established.

It is widely known that the engineer encounters great difficulty in the solution of problems using the methods of mathematical physics. One chapter, therefore, deals with the solution of actual problems with the omission of mathematical rigor. For this purpose several of the most simple problems in heat and mass transfer theory are investigated and described in detail to familiarize the engineer with the calculation techniques.

The last chapter is devoted to the solution of a number of problems where mass transfer takes place under the influence of several thermodynamic forces, such as, for example, the action of a pressure gradient. These cases of molecular-mass transfer of bound matter are of great practical interest for the solution of engineering problems involving diffusional processes in porous bodies.

The book contains a large number of original tables of basic solutions and parametric coefficients. Therefore, the book will be of general interest as well as specific interest for physicists and power engineers. As a result of the detailed and systematic presentation, the book can be used by research engineers who will be able to apply successfully the analytical methods for the solution of important problems of modern engineering.

Further development of the theory of transfer phenomena based on the thermodynamics of irreversible processes and its wide practical application in industry will contribute to the perfection of the technology of socialistic production.

The authors understand that this book, as any first attempt of this kind, contains a number of deficiencies. Therefore, all constructive suggestions by the reader, directed to improve further this book will be greatly appreciated.

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CHAPTER I

Basic Laws and Differential Equations of Molecular Transfer

Previous works of the authors have shown that all phenomena involving molecular transfer of heat, mass, and energy of ordered motion can be analytically described by the following law: the flux, which describes the rate of transfer of the corresponding entity, is directly proportional to the gradient of the volumetric concentration of this entity. The coefficient of proportionality (e.g. coefficient of energy conductivity) is equal to the coefficient of molecular diffusion. It is on the basis of this law that the corresponding differential transfer equations are derived (equations for the conductivity of heat, diffusion and motion of a viscous liquid).

These basic linear relationships, however, apply only to transfer processes of low intensity. In cases of high intensity the transfer phenomena should be evaluated in terms of reciprocal relationships. This means that heat transfer is influenced by transfer of mass and of energy of ordered motion. Therefore, the basic initial assumptions, on which the corresponding differential equations were based, have to be reviewed at the present time. For such basic changes in the analytical description of molecular transfer processes, a strict scientific method of investigation of transfer phenomena, and their interrelationships, is required.

Dutch and Belgian physicists (Onsager, Prigogine, de Groot) have developed a theory which is now known as the thermodynamics of irreversible processes. A special feature of this theory is the description of transfer phenomena of any entity on the basis of entropy transfer, which uniquely characterizes the complex reciprocal relationships of molecular transfer.

The differential equation for entropy transfer differs from the differential equation for energy transfer in that it contains an entropy source term, which results from the production of entropy in irreversible processes. The source of entropy is a function of the coordinates and of time. Therefore, the solution of differential equations such as those for heat conduction with variable sources and sinks, is of basic interest for the development of an analytic theory of the thermodynamics of irreversible processes.

In addition, the basic relationship of thermodynamic transfer theory (i.e. that the product of absolute temperature and rate of change of volumetric concentration of entropy is equal to the sum of the products of the various fluxes and their transfer driving forces) serves as the basic relationship in the determination of heat transfer potentials, mass transfer potentials and momentum transfer potentials in laminar fluid flow.

1. TRANSFER POTENTIALS

First of all, let us consider transfer potentials.

The following relationship is the basic axiom of the theory of potentials: when in thermodynamic equilibrium the potentials of a system of bodies, or the potentials in the various parts of the body, are equal:

$$\Pi_1 = \Pi_2. \tag{1.1}$$

The transfer of an entity (heat, electricity, mass) takes place in the direction from the larger to the smaller potential: $\Pi_1 > \Pi_2$.

The potential gradient is called a thermodynamic driving force: for example, the electrical potential gradient which is equal to the intensity of an electric field.

The quantitative measure of transfer of energy, dE, is equal to the product of the potential and the change of a generalized coordinate, dk, i.e., the transfer potential should satisfy the following relationship:

$$dE = \Pi\, dK. \tag{1.2}$$

Let us cite a few examples. During expansion of gas, when there is an increase in the volume, dV, the expansion work against the forces of the outside pressure, p, equals:

$$dW = pdV. \tag{1.3}$$

Consequently, the volume (K = V) is the generalized coordinate for the performance of mechanical work. The pressure ($\Pi = p$) will be the potential in this case.

The work of charging a conductor, dW, with an electrical potential φ ($\Pi = \varphi$) in an electrostatic field is equal to the product of the potential φ and the increase of the electric charge dq:

$$dW = \varphi\, dq. \tag{1.4}$$

The magnitude of the electrical charge ($K = q$) is in this case the generalized coordinate. In heat transfer processes the quantity of transferred heat dQ equals the product of temperature T (transfer potential of energy in the form of heat) and the increase in entropy dS (the generalized coordinate in this case is entropy):

$$dQ = TdS. \tag{1.5}$$

During transport of energy by means of mass transfer, the general coordinates are the mass of the matter, ν, measured in mols; and the transfer potential is a suitably selected chemical potential, μ, which in turn depends on the conditions of thermal and mechanical interaction of the system with the surrounding medium.

Let us dwell on this problem in more detail.

The Gibbs equation may be used to describe the first law of thermodynamics for open systems (systems where mass transfer takes place):

$$dQ = TdS = dE + pdV - \sum_k \mu_k \, d\nu_k \, , \tag{1.6}$$

where the subscript k denotes the number of the component taking part in mass exchange.

For a system where mechanical and thermal interaction with the surrounding medium takes place, there are four possible cases of interaction*:

a) Isochoric-isentropic interaction, $V = \text{const}, S = \text{const}$, ($dV = 0$; $dS = 0$), whence we get for the mass transfer potential the following expression:

$$\mu = \left(\frac{\partial E}{\partial \nu}\right)_{v,\, s} ; \tag{1.7}$$

b) Isobaric-isentropic interaction, $p = \text{const}$, $S = \text{const}$ ($pdV = dpV$; $dS = 0$). Let us denote the enthalpy of the system by H ($H = E + pV$), then from equation (1.6) we get:

$$\mu = \left(\frac{\partial F}{\partial \nu}\right)_{v,\, T} , \tag{1.8}$$

---

*It is assumed that all ν_k, with the exception of one, are constant.

c) Isochoric-isothermal interaction, $V =$ const, $T =$ const ($dV = 0$; $TdS = dTS$). In this case we get,

$$\mu = \left(\frac{\partial F}{\partial \nu} \right)_{v, T}, \tag{1.9}$$

where F is the free energy, $F = E - TS$;

d) Isobaric-isothermal interaction, $p =$ const, $T =$ const ($pdV = dpV$; $TdS = dTS$). Let us denote the free enthalpy by Φ, i.e.

$$\Phi = H - TS = E + pV - TS,$$

then from equation (1.6) we get

$$\mu = \left(\frac{\partial \Phi}{\partial \nu} \right)_{p, T}. \tag{1.10}$$

Consequently, the mass transfer potential is a function of the conditions of thermal and mechanical interaction of the system with the surrounding medium; this potential is equal to the derivative of a corresponding characteristic function ψ related to the mass:

$$\mu = \frac{\partial \psi}{\partial \nu}. \tag{1.11}$$

In the first case the characteristic function is the internal energy ($\psi = E$), in the second case — enthalpy ($\psi = H$) etc.

2. DIFFERENTIAL EQUATIONS FOR ENERGY, ENTHALPY AND MASS TRANSFER

In cases where the internal energy, E, volume of the system, V, with total mass, M, are calculated on the basis of unit mass (specific internal energy and volume) then the first law of thermodynamics applies to both a closed and an open system, i.e.

$$dq = de + pdv, \tag{2.1}$$

where $v = \frac{V}{M}$ —is the specific volume, and $e = \frac{E}{M}$ is the specific energy. For values of Q, E, V, related to the total mass of an open system the above equation assumes the following form:

$$dQ = dE + pdV - hdM, \tag{2.2}$$

where h is the specific enthalpy,

$$h = e + pv, \tag{2.3}$$

dQ is the heat given to the system $(dQ = Mdq)$.

Let us denote the fluxes of heat, mass, energy and enthalpy by j_q, j_m, j_e and j_h, respectively, that is

$$j_q = \frac{dQ}{Ad\tau}\ ;\ j_m = \frac{dM}{Ad\tau}\ ;\ j_e = \frac{dE}{Ad\tau}\ ;\ j_h = \frac{dH}{Ad\tau}\ , \tag{2.4}$$

where A is the area of an isopotential surface, i.e. a surface possessing the same transfer potential. For an isochoric process (V = constant) the fluxes of heat and mass transfer can be expressed on the basis of equation (2.2):

$$j_e = j_q + hj_m. \tag{2.5}$$

For an isobaric process (p = const) from equation (2.2) we get

$$j_h = j_q + hj_m. \tag{2.6}$$

The differential equation for energy transfer in a system with constant volume is the Umov equation for energy motion. It follows from the law of conservation of energy that the change of energy in a certain volume V, surrounded by a closed surface A, takes place as a result of transfer of energy through this surface, i.e.

$$\int\limits_{(V)} \frac{\partial E_v}{\partial \tau}\, dV = -\oint\limits_{(A)} (j_e)_n\, dA, \tag{2.7}$$

where E_v is the volumetric concentration of energy, τ is the time, and the subscript n denotes a normal to the isopotential surface.

Using the Ostrogradsky-Gauss transformation equation (2.7) yields the following differential equation for energy transfer:

$$\frac{\partial E_v}{\partial \tau} = -\nabla \cdot j_e, \tag{2.8}$$

where $\nabla \cdot$ denotes the symbol of divergence, i.e., $\nabla \cdot \equiv \text{div}$. Consequently, the local rate of change of the volumetric concentration of energy $\frac{\partial E_v}{\partial \tau}$ is directly proportional to the divergence of the vector flux of j_e.

The flux of energy, j_e is equal to the product of the volumetric density of energy, E_v, and the linear velocity of the molecular transfer w', i.e.:

$$j_e = E_v w', \tag{2.9}$$

For moving media, equation (2.8) is rewritten in the following form:

$$\frac{dE_v}{d\tau} = -\nabla \cdot E_v w' \tag{2.10}$$

where $\frac{d}{d\tau}$ is the substantial derivative,

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + w \,\text{grad} = \frac{\partial}{\partial \tau} + w\nabla,$$

w is the velocity of the macroscopic motion of the medium.

For an isobaric process, instead of the differential equation for energy transfer, we have an equation for enthalpy transfer:

$$\frac{\partial H_v}{\partial \tau} = -\nabla \cdot j_h, \tag{2.11}$$

where H_v is the volumetric concentration of enthalpy $\left(H_v = \frac{H}{V}\right)$.

The differential mass transfer equation for the component k can be written on the basis of the following assumption. In a certain volume, bounded by a control surface, the change of the volumetric concentration of the k-component of matter (ω_k) takes place as a result of transfer and action of a source (or sink) of the k-component of matter. Then on the basis of the law of conservation of mass and again using the Ostrogradskiy-Gauss transformation the differential equation for mass transfer is obtained:

$$\frac{\partial \omega_k}{\partial \tau} = -\nabla \cdot j_{mk} - \sum_r W_{kr}, \qquad (2.12)$$

where j_{mk} is the flux of mass of the k-component, and W_{kr} is the strength of the sink of the mass, related to the transition of the matter from the k-component to the r-component (occurrence of phase and chemical transformations).

The following equalities are applicable here on the basis of the law of conservation of mass:

$$W_{kr} + W_{rk} = 0; \quad W_{kr} = -W_{rk}. \qquad (2.13)$$

Summing up the system of equations (2.12) with respect to k, up to k = N, we get the differential equation for transfer of the entire mass of the system:

$$\frac{\partial \omega}{\partial \tau} = -\nabla \cdot \sum_{k=1}^{N} j_{mk}, \qquad (2.14)$$

where ω is the volumetric concentration of mass in the system.

$$\omega = \sum_{k=1}^{N} \omega_k. \qquad (2.15)$$

Chemical or phase transformations within the described volume do not change the entire mass of the system, i.e.

$$\sum_k \sum_r W_{kr} = 0. \qquad (2.16)$$

Consequently, the change of the total mass content of the system can take place only as a result of mass transfer.

3. HEAT AND MASS TRANSFER EQUATIONS IN DISPERSED SYSTEMS DURING PHASE TRANSFORMATION

Let us examine the transfer of heat and of bound matter in a dispersing medium within which phase transformations may occur.

The dispersing medium, e.g. a porous body, contains bound matter in the vapor state (subscript 1), liquid state (subscript 2) and solid state (subscript 3) — a three-phase system of bound matter.

The subscript 0 denotes the dispersing medium itself without the bound matter, e.g., the skeleton of a porous body.

The volumetric concentration of ω of the bound mass is equal to the product of the specific mass concentration and density of the medium γ_0, i.e.

$$\omega = u\gamma_0; \quad u = \sum_{k=1}^{3} u_k. \tag{3.1}$$

Local change of the bound mass in any phase may occur as a result of phase transformation du_{kr}, and as a result of transfer du_{kf}:

$$du_k = \sum_r du_{kr} + du_{kf}. \tag{3.2}$$

Equation (3.3) reflects the law of conservation of mass during transition from phase k to phase r, and the reverse:

$$du_{kr} = -du_{rk}. \tag{3.3}$$

Using differential equations, (2.12) and (2.14), for mass transfer, we get:

$$\gamma_0 \frac{\partial u_k}{\partial \tau} = -\nabla \cdot j_{mk} + \sum_r W_{rk}, \tag{3.4}$$

$$\gamma_0 \frac{\partial u}{\partial \tau} = -\nabla \cdot \sum_{k=1}^{3} j_{mk}, \tag{3.5}$$

where W_{rk} is the specific strength of the source of the bound matter of phase k (transformation of mass from phase r to phase k):

$$W_{rk} = \gamma_0 \frac{\partial u_{rk}}{\partial \tau}. \tag{3.6}$$

The differential equation for heat transfer with isobaric interaction of the system with the surroundings may be obtained from equation (2.11). The local rate of change of the volumetric concentration of enthalpy, γh, of the system (the dispersing medium and the mass bound in it in three phases) equals the divergence of the flux of enthalpy. The latter, from equation (2.6) equals the sum of the heat flux and the product of the specific enthalpy and flux of mass:

$$\frac{\partial}{\partial\tau}\left(\gamma_0 h_0+\sum_{k=1}^{3}\gamma_0 h_k u_k\right)=-\nabla\cdot\left(j_q+\sum_{k=1}^{3}h_k j_{mk}\right). \tag{3.7}$$

Let us denote the specific heat capacity at constant pressure by c_k, i.e.:

$$c_k=\frac{dh_k}{dt};\quad c_0=\frac{dh_0}{dt}, \tag{3.8}$$

where $t=T-273$ is the temperature in degrees centigrade, and c is the total heat capacity:

$$c=c_0+\sum_{k=1}^{3}c_k u_k. \tag{3.9}$$

Then, from the equation for enthalpy we get the equation for heat transfer:

$$c\gamma_0\frac{\partial t}{\partial\tau}=-\nabla\cdot j_q+\sum_{k=1}^{3}\sum_{r=1}^{3}h_k W_{kr}. \tag{3.10}$$

It is of interest to examine the special case of two phases (k=1.2) of the bound mass (vapor and liquid) corresponding to the region of positive temperatures on the centigrade scale. In this case the heat transfer equation is:

$$c\gamma_0\frac{\partial t}{\partial\tau}=-\nabla\cdot j_q+\rho W, \tag{3.11}$$

where ρ is the specific heat of vaporization $(\rho=\rho_{21}=\rho_{12})$, equal to the difference of enthalpies $(\rho_{12}=h_1-h_2)$, and W is a source, or sink, of bound mass, $(W = W_{12} = -W_{21})$. In cases where the transfer of liquid and vapor takes place under the action of capillary and hydrodynamic forces (transfer of mass at a linear velocity w_k), an additional term must be added to the right-hand side of equation (3.11):

$$-\sum_{k=1}^{2}c_k\gamma_k w_k\nabla t.$$

$$\frac{\gamma_0}{1-b}\,\frac{\partial u_2}{\partial \tau} = -\nabla\cdot(j_1+j_2). \tag{3.20}$$

In cases where the sublimation ($W_{31}=0$) and desublimation ($W_{13}=0$), can be neglected, the heat transfer equation is considerably simplified and assumes the following form:

$$c\gamma_0\frac{\partial t}{\partial \tau} = -\nabla\cdot j_q + (h_1-h_2)W_{12} + $$

$$+(h_2-h_3)\frac{b\gamma_0}{1-b}\,\frac{\partial u_2}{\partial \tau} - \sum_{k=1}^{2} c_k\gamma_k w_k \nabla t. \tag{3.21}$$

The first term on the right-hand side of the above equation describes heat transfer by conduction. The second term is a source (or sink) of heat resulting from condensation of vapor (or vaporization of liquid), and the third term is a source of heat, resulting from the transformation of water into ice. Finally, the fourth term corresponds to the convective heat transfer by vapor and liquid streams. In order to express the system of differential transfer equations in a solvable form, it is necessary to express the fluxes of heat and matter in terms of thermodynamic driving forces. This can be accomplished on the basis of modern concepts about molecular energy and mass transfer with the methods of the thermodynamics of irreversible processes.

4. BASIC RELATIONSHIPS OF THE THERMODYNAMICS OF IRREVERSIBLE PROCESSES

Underlying the thermodynamics of irreversible processes are two principles: the linear law and the Onsager reciprocal relationships. In accordance with the linear law, the velocity, v, at which the system approaches equilibrium, is proportional to the thermodynamic driving force X, which can be expressed in terms of a potential gradient (the flux of electricity is proportional to the voltage gradient or field intensity, the flow of heat in heat transfer by conduction is proportional to the temperature gradient, the diffusion flux is proportional to the concentration gradient, etc.).

The linear law is a generalization of empirical relationships which have been established for a system whose state approaches equilibrium.

Consequently, the linear law is an experimental law and does not have a strict theoretical basis. It should be noted, however, that the linear law may be derived from the statistical theory of fluctuations, if the state of the system is close to equilibrium.

In the presence of several driving forces, the rate v_i at which the system approaches equilibrium, in accordance with the linear law, can be expressed in the form of the following summation:

$$v_i = \sum_j L_{ij} X_j. \tag{4.1}$$

The diagonal terms characterize the direct effect, corresponding to the basic driving force, and the cross-terms correspond to the influence of the subsequent, induced phenomena.

Some applications of equation (4.1) have been known for a long period of time. However, the general formulation of the linear law has been given for the first time in the thermodynamics of irreversible processes. The thermodynamic driving force is determined from a basic relationship of the thermodynamics of irreversible processes; the product of the rate of entropy production $\dot{S}$ and the absolute temperature is equal to the sum of the products of the driving forces and the velocity of approach of the system to equilibrium:

$$\dot{S}T = \sum_i X_i v_i. \tag{4.2}$$

Using equation (4.1), we get

$$\dot{S}T = \sum_{i,j} L_{ij} X_i X_j. \tag{4.3}$$

Since $S > 0$, the summation in equation (4.3) is actually a positive quantity.

Another principle of thermodynamics of irreversible processes is the reciprocal relationship

$$L_{ij} = L_{ji}. \tag{4.4}$$

The above states that the cross-kinetic coefficients L are equal.

This relationship was demonstrated by Onsager on the basis of statistical mechanics.

The principle of reciprocity results from the property of microscopic reversibility, i.e., under conditions of equilibrium the average rates of any molecular process are equal in all directions.

The above considerations were used for the derivation of the formal theory of the rate of irreversible processes; this theory has been successfully applied to transfer phenomena and to chemical reaction kinetics. The basic transfer equation is the differential equation for entropy change which is different from the Umov differential equation of energy change by having an additional term expressing the positive entropy source (entropy producting in irreversible processes). Because the laws of classical thermodynamics can be based on and derived by means of the methods of statistical mechanics, some researchers tend to consider classical thermodynamics a part of statistical mechanics. The molecular-kinetic theory of heat not only provides an insight into the physical nature of the basic thermodynamic relationships, but it also augments the conclusions and results of classical thermodynamics. The successful combination of these two methods permits more profound and detailed study of a number of physical and physico-chemical phenomena.

5. THE SYSTEM OF ONSAGER LINEAR EQUATIONS

The fluxes are denoted by I (the quantity of substance transferred during a unit of time through a unit of area of isopotential surface), the thermodynamic driving forces by X and finally the kinetic coefficients by L. Thus the molecular transfer of electrical energy, heat and mass in a homogeneous medium can be described by the following system of linear equations:

$$\left.\begin{aligned} I_1 &= L_{11}X_1 + L_{12}X_2 + L_{13}X_3 \\ I_2 &= L_{21}X_1 + L_{22}X_2 + L_{23}X_3 \\ I_3 &= L_{31}X_1 + L_{32}X_2 + L_{33}X_3 \end{aligned}\right\}. \tag{5.1}$$

With the above equations the reciprocal relationships of Onsager apply:

$$L_{12} = L_{21};\ L_{13} = L_{31};\ L_{23} = L_{32}. \tag{5.2}$$

The thermodynamic driving force for heat flow is the ratio of the temperature gradient to the temperature, i.e.,

$$X_1 = X_q = -\frac{\nabla T}{T}. \tag{5.3}$$

Analogous relationships apply to the driving forces for mass and electricity:

$$X_2 = X_м = -T\nabla\left(\frac{\mu}{T}\right), \tag{5.4}$$

where μ is the chemical potential (the derivative of thermodynamic functions during isochoric-isothermal interaction with respect to the mass of the transferred matter).

$$X_3 = X_c = -\nabla\varphi, \tag{5.5}$$

where, φ is the electrical potential.

A.V. Lykov and A.P. Porkhayev have established that heat and mass transfer in a boundary layer are influenced by a hydrodynamic driving force during the flow of a heated gas around a moist body. In this case it is expedient to consider the phenomenon of internal friction or viscosity as a process of molecular transfer of momentum. Usually the viscosity of a liquid, or gas, is considered as representative of the transfer of a certain amount of momentum. In this case, Newton's Law of viscous motion cannot be compared with the law of conductivity and diffusion in the system of linear Onsager equations, because the driving forces of heat, mass and momentum transfer represent tensors of different rank. On the basis of the Curie theorem, certain driving forces can be included in the Onsager system. These forces represent tensors of equal rank, or ranks differing by an even number. This condition is not satisfied by driving forces of heat and momentum. However, if viscosity is associated with the molecular transfer of the kinetic energy of ordered motion, then contradictions of the Curie theorem no longer exist, because the driving forces for the transfer of heat, mass, and the kinetic energy of ordered motion are tensors of one and the same rank.

The driving force of molecular transfer of kinetic energy of laminar flow is the gradient of kinetic energy per unit mass of liquid, i.e.,

$$X_4 = X_9 = -\nabla\left(\frac{\gamma w^2}{2}\right), \tag{5.6}$$

where γ and w are the density and linear velocity of the fluid flow, respectively.

In this case another equation has to be added to the system of linear equations (5.1):

$$I_4 = L_{41}X_1 + L_{42}X_2 + L_{43}X_3 + L_{44}X_4, \tag{5.7}$$

where I_4 is the flux of kinetic energy of laminar flow, transferred as a result of random molecular motion.

Equations (5.3) to (5.6) show that the thermodynamic driving force of the corresponding form of transfer is directly proportional to the potential gradient of this kind of transfer. The minus sign is necessary as a result of a basic axiom of transfer: the transfer takes place from the higher potential to the lower potential. If we denote the transfer potential i by Π_i, then we can write

$$X_i = -K_i \nabla \Pi_i. \tag{5.8}$$

The coefficient K_i is a function of the method by which fluxes and forces are selected; in this specific case it can be taken to equal one. In the thermodynamics of irreversible processes it is shown that the reciprocal relationship of Onsager can be applied with any method of selection of fluxes and forces; i.e. it is shown that there exists an invariance of the symmetry of the matrices, L_{ik}.

Thus, in the general case, the molecular transfer is described by the Onsager system of linear equations:

$$\left.\begin{array}{l} I_1 = \sum_i L_{1i} X_i \\ I_2 = \sum_i L_{2i} X_i \\ I_3 = \sum_i L_{3i} X_i \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ I_j = \sum_i L_{ji} X_i \end{array}\right\}. \tag{5.9}$$

Depending on the form of the transfer, the choice of thermodynamic forces is limited to only a few.

6. DIFFERENTIAL EQUATION OF ENTROPY TRANSFER

The basic relationship for the selection of thermodynamic forces (the product of absolute temperature and the rate production of volumetric concentrations of entropy is equal to the sum of the products of the fluxes and the thermodynamic forces is the differential equation for entropy transfer.

Four differential equations are necessary to set up the entropy balance.

1. The differential equation for mass transfer, obtained on the basis of the law of mass conservation.

Let us denote the concentration of the k-component of the bound matter by ω. (M_k is the mass of the k-component per unit of total volume V). The rate of the chemical reaction is denoted by W_c (the quantity of bound matter, formed as a result of chemical reactions in a unit volume during a unit of time). Then we have:

$$\frac{\partial \omega_k}{\partial \tau} = -\nabla \cdot \omega_k \vec{w}_k + \nu_k W_c, \tag{6.1}$$

where $\vec{w}_k$ is the vector velocity of component k, and

ν_k is a quantity equal to the stoichiometric number of this component multiplied by the molecular weight. The following relationship also applies:

$$\sum_k \nu_k = 0. \tag{6.2}$$

Summing up equation (6.1) with respect to all components k, we get:

$$\frac{\partial \omega}{\partial \tau} = -\nabla \cdot \omega \vec{w}, \tag{6.3}$$

where ω is the total concentration or the density of the mixture, i.e.,

$$\omega = \sum_k \omega_k = \frac{1}{V} \sum_k M_k = \frac{M}{V}, \tag{6.4}$$

$\vec{w}$ is the velocity vector of the center of gravity of the mass M, i.e.,

$$\vec{w} = \frac{1}{\omega} \sum_k \omega_k \vec{w}. \tag{6.5}$$

Let us denote the substantial derivative with respect to time by $\frac{d}{d\tau}$, i.e.,

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \vec{w} \nabla. \tag{6.6}$$

The flow of component k is determined with relation to the moving center of gravity of the entire mass:

$$I_k = \omega_k \left(\vec{w}_k - \vec{w}\right). \tag{6.7}$$

From equations (6.5) and (6.7) it can be deduced that

$$\sum_{k=1}^{n} I_k = 0. \tag{6.8}$$

In this case the differential equation of transfer of the k-component can be written

$$\frac{d\omega_k}{d\tau} = -\omega_k \nabla \cdot \vec{w} - \nabla \cdot I_k + \nu_k W_c. \tag{6.9}$$

When the weight concentration, or specific mass content, is introduced, then

$$u_k = \frac{\omega_k}{\omega} = \frac{M_k}{M}, \tag{6.10}$$

and the differential equation (6.9) will assume the following form:

$$\omega \frac{du_k}{d\tau} = -\nabla \cdot I_k + \nu_k W_c. \tag{6.11}$$

On the basis of equation (6.6), equation (6.3) may be rewritten as

$$\frac{d\omega}{d\tau} = -\omega \nabla \cdot w \tag{6.12}$$

2. The second differential equation concerns the action of external forces. Let us denote pressure by P, and the external force acting on a unit of mass of the k-component by F_k. Then, neglecting the forces of viscosity, the following differential equation can be written:

$$\omega \frac{d\vec{w}}{d\tau} = -\nabla P + \sum_{k=1}^{n} \vec{F}_k \omega_k. \qquad (6.13)$$

3. The differential equation for energy is:

$$\frac{\omega d\left(\frac{1}{2}\vec{w}^2 + E\right)}{d\tau} = -\nabla\cdot(P\vec{w} + I_q) + \sum_{k=1}^{n} \vec{F}_k \vec{w}_k \omega_k, \qquad (6.14)$$

where E is the internal energy, and
I_q is the heat flux.

The above equation can be rewritten in a different form. Let us introduce the concept of energy per unit volume

$$E_{0\,\mathrm{vol}} = E_v + \frac{1}{2} w^2.$$

then,

$$\frac{\partial E_{o\,\mathrm{vol}}}{\partial\tau} = -\nabla\cdot I_E, \qquad (6.15)$$

where,

$$I_E = (E_{o\,\mathrm{vol}} + Pv)\,\omega\,\vec{w} + I_q, \qquad (6.16)$$

and v is the specific volume.

4. The fourth equation is the Gibbs equation

$$T\frac{dS}{d\tau} = \frac{dE}{d\tau} + P\frac{dV}{d\tau} - \sum_k \mu_k \frac{dM_k}{d\tau}, \qquad (6.17)$$

expressed in terms of total derivatives with respect to the center of gravity of the moving mass. This equation can be applied both to closed as well as open systems.

Let us rewrite the above equations in terms of specific values:

$$T\frac{ds}{d\tau}=\frac{de}{d\tau}+p\frac{dv}{d\tau}-\sum_k \mu_k\frac{du_k}{d\tau}, \tag{6.18}$$

where, $s=\frac{S}{M}$ is the specific entropy, and e is the specific energy.

The mass M is a variable quantity and the following relationship applies:

$$Ts=e+pv-\sum_k \mu_k u_k \ . \tag{6.19}$$

The average chemical potential is equal to

$$\mu=\sum_k \mu_k u_k=\frac{1}{\omega}\sum_k \mu_k \omega_k=e-Ts+pv. \tag{6.20}$$

If equation (6.13) is multiplied by w and subtracted from equation (6.14) then the kinetic energy of motion of the center of gravity is eliminated and an expression for the change of energy is obtained:

$$\omega\frac{de}{d\tau}=-p\nabla\cdot w-\nabla\cdot j_q+\sum_k \vec{F}_k j_k \ . \tag{6.21}$$

Substituting here equation (6.12), we get

$$\frac{de}{d\tau}=-p\frac{dv}{d\tau}-\frac{1}{\omega}\nabla\cdot j_q+\frac{1}{\omega}\sum_k \vec{F}_k j_k \ . \tag{6.22}$$

Using equations (6.11) to (6.22), equation (6.18) yields

$$\omega T\frac{ds}{d\tau}=-\nabla\cdot j_q+\sum_k \vec{F}_k j_k+\sum_k \mu_k\nabla\cdot j_k-W_c\sum_k \mu_k\nu_k \ . \tag{6.23}$$

The above may be written as

$$\omega\frac{ds}{d\tau}=-\nabla\cdot j_s+\sigma, \tag{6.24}$$

where

$$j_s = \frac{1}{T}\left(j_q - \sum_k \mu_k j_k\right), \tag{6.25}$$

$$\sigma = \frac{1}{T}\left(j_q X_q + \sum_k j_k X_k + A W_c\right), \tag{6.26}$$

$$X_q = -\frac{1}{T}\nabla T; \quad X_k = \vec{F}_k - T\nabla\left(\frac{\mu_k}{T}\right),$$

$$A = -\sum_k \mu_k \nu_k \text{ — chemical affinity} \tag{6.27}$$

Consequently, the local rate of change of entropy is equal to the divergence of the flow of entropy, j_s plus a source of entropy σ, resulting from the irreversibility of heat and mass transfer.

Let us denote the volumetric entropy concentration by S_v.

$$S_v = \frac{S}{V} = s\omega. \tag{6.28}$$

Then, using the following relationship

$$\omega \frac{ds}{d\tau} = \frac{\partial S_v}{\partial \tau} + \nabla \cdot S_v w, \tag{6.29}$$

we will get the differential equation for entropy transfer in the following form:

$$\frac{dS_v}{d\tau} = -\nabla \cdot (j_s + S_v w) + \sigma. \tag{6.30}$$

7. SYSTEM OF DIFFERENTIAL EQUATIONS FOR HEAT AND MASS TRANSFER IN POROUS BODIES

Let us consider in detail the transfer of heat and mass in porous bodies with phase transformations. We shall consider the motion of liquid and vapor inside a porous body ($u_3 = 0$). At atmospheric pressure the specific mass of vapor in the pores of the body is insignificantly small in comparison with the specific mass of the liquid $(u_2 \gg u_1)$. We shall therefore assume that the total specific mass content of the bound matter (moisture) is equal to the specific mass content of the liquid. This assumption applies to the method of determining the moisture content of a body (drying in a thermostat at 105°C to constant weight), by removal of the moisture in liquid form.

The differential transfer equation for the k-phase of the bound matter in a porous body, under conditions of absorption or desorption, assumes the following form:

$$\gamma_0 \frac{\partial u_k}{\partial \tau} = -\nabla \cdot j_{mk} + \sum_i \varepsilon_{ki} \frac{\partial u_k}{\partial \tau} \gamma_0. \tag{7.1}$$

The second term of the right-hand side of equation (7.1) must be considered as a source, or sink, of the k-phase of bound matter, resulting from its transformation from phase i to phase k, or from phase k to phase i.

Using equation (3.3), the differential equation for mass transfer of phase k can be rewritten as follows:

$$\gamma_0 \frac{\partial u_k}{\partial \tau} = -\nabla \cdot j_{mk} + \sum_k \varepsilon_{ki} \frac{\partial u_i}{\partial \tau} \gamma_0. \tag{7.2}$$

The differential equation for heat transfer is obtained from the equation for enthalpy transfer. The latter can be formulated on the basis of the law of conservation of energy: the local rate of enthalpy change in an infinitesimally small volume is equal to the divergence of the enthalpy flux:

$$\gamma_0 \frac{\partial}{\partial \tau}\left(h_0 + \sum_k h_k u_k\right) = -\nabla \cdot \left(j_q + \sum_k h_k j_{mk}\right). \tag{7.3}$$

In cases where the body contains only liquid and vapor, $(u_3 = 0)$, then $u = u_2$, because the mass of moist gas at atmospheric pressure in pores of the body is insignificantly small in comparison with the mass of the liquid $(u_1 \ll u_2)$.

The transfer of the bound matter in any state, is determined by the action of the mass driving force (the gradient of the potential of moisture transfer) and by the thermal driving force (temperature gradient) i.e.:

$$j_{mk} = -\lambda'_k \nabla \theta_k - \lambda_k \nabla t, \tag{7.4}$$

where θ_k is the transfer potential of the bound matter in the k-phase;
λ_k is the ordinary coefficient of mass transfer of the bound matter, and
λ_k' is the thermal coefficient of mass transfer of the bound matter.

At isothermal conditions ($\Delta t = 0$) the transfer of the bound matter in the form of vapor is determined only by the first term of equation (7.4). The potential of the diffusion transfer of vapor is the partial pressure of vapor, or the relative concentration of the molecules, i.e.:

$$j_1 = -\lambda_1' \nabla p_1 = -\lambda_1' P \nabla \nu, \qquad (7.5)$$

where $P = p_1 + p_b$ is the total pressure;
p_b is the partial pressure of air; and
ν the relative concentration of the molecules equal to the ratio of volumetric concentration of vapor molecules (n_1) to the volumetric concentration of the molecules of the mixture ($n_1 + n_b$),

$$\nu = \frac{n_1}{n_1 + n_b}. \qquad (7.6)$$

Using the equation of state for an ideal gas:

$$p = nkT = \frac{\omega RT}{\mu}, \qquad (7.7)$$

where μ is the molecular weight,
k is the Boltzman constant,
R is the universal gas constant, and
ω is the volumetric concentration of gas or vapor; formula (7.5) is transformed into the usual form of Fick's diffusion law:

$$j_1 = -\lambda_1' \frac{RT}{\mu_1} \nabla \omega_1 = -D' \nabla \omega_1, \qquad (7.8)$$

where, $D' = \frac{\lambda_1' RT}{\mu_1}$ is the diffusion coefficient of the vapor in air.

In cases of non-isothermal molecular transfer of vapor it is necessary to use formula (7.4).

An analogous relationship can be written for mass transfer in the form of liquid:

$$j_2 = -\lambda_2' \nabla \theta_2 - \lambda_2'' \nabla t. \tag{7.9}$$

During isothermal conditions, the transfer of liquid is described by

$$j_2 = -\lambda_2' \nabla \theta_2. \tag{7.10}$$

At constant temperature the transfer potential of liquid, θ_2 is a simple function of the mass concentration in the liquid, u_2 (moisture concentration in the body, u).

The magnitude of the vector $\nabla \theta_2$ is equal to $\partial\theta_2/\partial n$, where n is the normal to the isopotential surface. Now we have:

$$\frac{\partial\theta_2}{\partial n} = \left(\frac{\partial \theta_2}{\partial u}\right)_T \frac{\partial u}{\partial n}. \tag{7.11}$$

The partial derivative of moisture content of the body with respect to the potential of liquid transfer at constant temperature is equal to the isothermal mass capacity:

$$c_T' = \left(\frac{\partial u}{\partial \theta_2}\right)_T. \tag{7.12}$$

The potential gradient θ_2 is then equal to:

$$\nabla\theta_2 = \frac{1}{c_T'} \nabla u. \tag{7.13}$$

Consequently, the liquid transfer can be expressed as:

$$j_2 = -\frac{\lambda_2'}{c_T'} \nabla u, \tag{7.14}$$

that is, the flux of liquid is proportional to the gradient of the specific moisture content (the law of isothermal moisture conductivity).

Let us multiply and divide the right-hand side of equation (7.14) by the density of an absolutely dry body, γ_0:

$$j_2 = -a_2'\gamma_0 \nabla u; \quad a_2' = \frac{\lambda_2'}{c_T'\gamma_0}. \tag{7.15}$$

The transfer coefficient, a_2', analogous to the coefficient of thermal diffusivity, a, will be called "the mass exchange coefficient of potential conductivity;" since the coefficient, a, may be called "the heat exchange coefficient of potential conductivity."

The law of molecular heat transfer (the Fourier law) can be written in an analogous form:

$$q = -\lambda \nabla T = -a\gamma_0 \nabla h; \quad a = \frac{\lambda}{c\gamma_0}, \tag{7.16}$$

because

$$\nabla T = \left(\frac{\partial T}{\partial h}\right)_{u,p} \nabla h = -\frac{1}{c_p} \nabla h, \tag{7.17}$$

where h is the enthalpy of a moist body, calculated with respect to one kilogram of mass of an absolutely dry body (kcal/kg of dry body), and

c_p is the specific heat capacity at constant pressure of a moist body, calculated with respect to one kilogram of an absolutely dry body (kcal/degree-kg of a dry body).

Thus,

$$c = c_0 + c_2 u. \tag{7.18}$$

It should be noted that the expression for the isothermal moisture conductivity applies only under conditions of a continuous moisture-content field. In cases where we have a system of separate but touching bodies with different specific mass capacities (moisture capacities), at the boundaries of contact there will exist step-changes in the moisture content. In effect, the transfer of bound matter (moisture) through the surfaces of contact can take place from a smaller moisture content to a larger, and then the moisture content cannot be considered a potential of moisture transfer.

During non-isothermal conditions it is necessary to use formula (7.4). Moreover, in the general case the transfer potential θ_2 is a function of both moisture content and temperature. In this case the basic transfer equation for liquid is

$$j_2 = -\lambda_2' \nabla \theta_2 - \lambda_2'' \nabla t = -\lambda_2' \left(\frac{\partial \theta_2}{\partial u}\right)_T \nabla u - \\ -\lambda_2' \left(\frac{\partial \theta_2}{\partial T}\right)_u \nabla t - \lambda_2'' \nabla t. \tag{7.19}$$

The quantity $\left(\frac{\partial \theta_2}{\partial T}\right)_u$ is the temperature coefficient of the potential of liquid transfer, and $\left(\frac{\partial \theta_2}{\partial u}\right)_T$ is the reciprocal of the specific isothermal mass capacity.

Let us rewrite formula (7.19) taking into consideration equations (7.13) and (7.15):

$$j_2 = -\frac{\lambda_2'}{c_T'} \nabla u - \frac{\lambda_2'}{c_T'} c_T \left[\left(\frac{\partial \theta_2}{\partial T}\right)_u + \frac{\lambda_2''}{\lambda_2'}\right] \nabla t. \tag{7.20}$$

We can now introduce the thermal gradient coefficient

$$\delta_2 = -\frac{\nabla u}{\nabla t} = -\frac{\Delta u}{\Delta t}, \qquad j_2 = 0, \tag{7.21}$$

i.e., the thermal gradient coefficient is numerically equal to the ratio of the moisture content and temperature gradients in a stationary state and in the absence of mass transfer. Then the coefficient can be written as

$$\delta_2 = c_T' \left(\frac{\partial \theta_2}{\partial T}\right)_u + c_T' \frac{\lambda_2''}{\lambda_2'}. \tag{7.22}$$

In cases where the quantity $\frac{\lambda_2''}{\lambda_2'}$ is small, the thermal gradient coefficient, δ_2, is equal to the temperature coefficient of the liquid transfer potential multiplied by the isothermal specific mass capacity of the body.

If, however, the temperature coefficient is small,

$$\left(\frac{\partial\theta_2}{\partial T}\right)_u \ll \frac{\lambda_2''}{\lambda_2'}, \tag{7.23}$$

then the thermal gradient coefficient δ_2 is equal to the product of the Soret coefficient, δ_s, and the specific mass capacity:

$$\delta_2 = \delta_s c_T' = \frac{\lambda_2''}{\lambda_2'} c_T. \tag{7.24}$$

Consequently, the equation for liquid transfer (7.20) can be written as

$$j_2 = -a_2'\gamma_0 \nabla u - a_2'\gamma_0\delta_2 \nabla t. \tag{7.25}$$

Let us substitute equation (7.25) into equation (7.2):

$$\gamma_0 \frac{\partial u}{\partial \tau} = \nabla\cdot(a_2'\gamma_0 \nabla u + a_2'\delta_2 \nabla t) + \varepsilon\gamma_0 \frac{\partial u}{\partial \tau}, \tag{7.26}$$

where $\varepsilon = \varepsilon_{21} = \varepsilon_{12}$ is the phase transformation number for liquid into vapor and the reverse.

The quantity $\varepsilon\gamma_0 \frac{\partial u}{\partial \tau}$ represents a source, or sink, of liquid as a result of vaporization or condensation. The coefficients a_2' and δ_2 are variables; they are functions of temperature and moisture content of the body. If the calculations are conducted by zones, into which the process of unsteady heat and mass transfer is divided, then for every zone the coefficients a_2' and δ_2 can be considered constant. In this case equation (7.26) may be expressed in the following form:

$$\frac{\partial u}{\partial \tau} = a_2' \nabla^2 u + a_2'\delta_2 \nabla^2 t + \varepsilon \frac{\partial u}{\partial \tau}. \tag{7.27}$$

By introducing the equivalent coefficient for liquid transfer (coefficient of potential conductivity of transfer of the bound matter):

$$a' = \frac{a_2'}{1-\varepsilon}, \tag{7.28}$$

the differential equation (7.27) for mass transfer assumes the following form:

$$\frac{\partial u}{\partial \tau} = a' \nabla^2 u + a'\delta_2 \nabla^2 t. \tag{7.29}$$

This relationship applies for $\varepsilon < 1$. When the phase transformation number is equal to one there will be no motion of the liquid and the change in moisture content will take place only as a result of phase transformations.

In this case the mass transfer should be considered on the basis of relationships describing the motion of vaporized moisture.

Let us dwell on this problem in a little more detail.

The differential equation (3.5) has in this case the following form:

$$\gamma_0 \frac{\partial u}{\partial \tau} = -\nabla \cdot (j_1 + j_2). \tag{7.30}$$

Let us substitute for j_1 and j_2 their corresponding expressions from equation (7.4). Then, differential equation (7.30) is:

$$\frac{\partial u}{\partial \tau} = \nabla \cdot [(a_1' + a_2') \nabla u + (a_1'\delta_1 + a_2'\delta_2) \nabla t]. \tag{7.31}$$

It is assumed here that vectors j_1 and j_2 are parallel. By introducing the following notation:

$$a' = a_1' + a_2'; \quad \delta = \frac{a_1'\delta_1 + a_2'\delta_2}{a_1' + a_2'}, \tag{7.32}$$

equation (7.31) can be rewritten:

$$\frac{\partial u}{\partial \tau} = \nabla \cdot (a' \nabla u + a' \delta \nabla t). \tag{7.33}$$

In this case the transformation number is given by:

$$\varepsilon = \frac{j_1}{j_1 + j_2}. \tag{7.34}$$

In cases where bound matter is transferred in the vapor form ($\varepsilon = 1$, $j_2 = 0$), equation (7.31) assumes the following form:

$$\frac{\partial u}{\partial \tau} = \nabla \cdot (a_1' \nabla u + a_1' \delta_1 \nabla t). \tag{7.35}$$

It is of interest to compare equations (7.33) and (7.29). We shall determine j_1 from equation (7.34) and substitute the resulting expression into equation (7.30):

$$\gamma_0 \frac{\partial u}{\partial \tau} = -\nabla \cdot \left(\frac{j_2}{1-\varepsilon}\right) = \nabla \cdot \left[\frac{a_2'}{1-\varepsilon}(\nabla u + \delta_2 \nabla t)\right]. \tag{7.36}$$

Assuming that ε is constant, we find that:

$$a' = \frac{a_2'}{1-\varepsilon}; \quad \delta = \delta_1 = \delta_2. \tag{7.37}$$

The above equality applies only when $u_2 = u$, because $u_2 \gg u_1$.

In the derivation of the differential equation for heat transfer we considered that vapor in the pores of the body is in thermal and molecular equilibrium with the liquid. The mass transfer takes place slowly, and in such a manner that the temperature of the liquid is equal to the temperature of the walls of the skeleton of the porous body.

The differential equation for heat transfer is obtained from equation (3.11), where the source (sink) of the mass is determined by equation (3.15):

$$c\gamma_0 \frac{\partial t}{\partial \tau} = \nabla \cdot (\lambda \nabla t) + \varepsilon \rho \frac{\partial u}{\partial \tau} - \sum_{k=1}^{2} c_k \gamma_k w_k \nabla t, \tag{7.38}$$

where

$$c\gamma_0 = (c_0 + c_2 u_2 + c_1 u_1)\gamma_0 = (c_0 + c_2 u_2)\gamma_0. \tag{7.39}$$

Because the specific mass content of the vapor is insignificant in comparison with the mass content of the liquid ($(u_1 \ll u_2)$, then $u_2 = u$.

Equation (7.38) resembles the Fourier-Kirchhoff equation for heat transfer in flowing liquids, to which it reduces when $\varepsilon = 0$; $w_1 = 0$; $c = c_2$ and $\gamma_0 = \gamma_2$.

In equation (7.38), the quantity $\sum_k c_k \gamma_k w_k \nabla t$ accounts for the convective contribution to heat transfer, by moving liquid and vapor; the first term of the right-hand side of this equation $[\nabla\cdot(\lambda\nabla t)]$ represents the conductive contribution to heat transfer (heat transfer by conduction).

The quantity $\varepsilon\rho\gamma_0\frac{\partial u}{\partial\tau}$ represents the source (sink) of heat, resulting from condensation (vaporization) of the vapor.

In the heating of moist bodies, or in drying, the heat transfer by liquid or vapor motion (the convective contribution) is very small in comparison with the heat source of phase transformation, and in comparison with the conductive contribution to heat transfer. Therefore, these terms can be neglected if the Reynolds number for the moving vapor or liquid is less than unity, that is, if

$$\mathrm{Re}=\frac{j_k d_э}{\eta}<1, \tag{7.40}$$

where η is dynamic viscosity of the bound matter in motion as vapor or liquid, in kg/m-hr;

$j_k=\gamma_k w_k$ is the molar flux of the bound matter, and

$d_э$ is the equivalent diameter of the porous body, in meters.

In drying processes the Reynolds number for the moving vapor or liquid is usually less than 0.1.

In view of the above remarks, the equation for heat transfer can be reduced to the Fourier equation for heat conduction with a heat source or sink resulting from phase transformations, i.e.,

$$\frac{\partial t}{\partial\tau}=\frac{1}{c\gamma_0}\nabla\cdot(\lambda\nabla t)+\frac{\varepsilon\rho}{c}\frac{\partial u}{\partial\tau}. \tag{7.41}$$

The system of differential equations of mass and heat transfer (7.26) and (7.41) is a non-linear system, because the transfer coefficients are functions of both the moisture content and temperature. Calculations of unsteady process of heat and mass transfer can be made assuming constant transfer coefficient zones (average values can be taken for every zone). A considerably simplified system of equations can thus be used and good agreement with experimental values are obtained. Consequently, for a zonal system of calculating the processes of heat and mass transfer, the system of differential equations assumes the following form:

$$\frac{\partial t}{\partial\tau}=a\nabla^2 t+\varepsilon\frac{\rho}{c}\frac{\partial u}{\partial\tau}, \tag{7.42}$$

$$\frac{\partial u}{\partial\tau}=a'\nabla^2 u+a'\delta\nabla^2 t. \tag{7.43}$$

The differential equation for heat conduction (7.42) can be made similar to equation (7.43) if instead of the temperature we introduce the enthalpy (specific heat content) of the moist body, i.e.,

$$\frac{\partial h}{\partial \tau} = a\nabla^2 h + \varepsilon\rho \frac{\partial u}{\partial \tau}. \qquad (7.44)$$

Conversely, the differential equation for mass conduction (7.43) can be rewritten in a form similar to the equation for heat conduction (7.42):

$$\frac{\partial \theta_2}{\partial \tau} = a' \nabla^2 \theta_2 + a' \frac{\delta}{c'_T} \nabla^2 t. \qquad (7.45)$$

If the potential of liquid transfer θ_2 is written without the subscript 2, and if equation (7.42) is taken into account, then the system of differential equations for heat and mass transfer are:

$$\frac{\partial t}{\partial \tau} = a \nabla^2 t + \varepsilon\rho \frac{c'_T}{c} \frac{\partial \theta}{\partial \tau}, \qquad (7.46)$$

$$\frac{\partial \theta}{\partial \tau} = a' \nabla^2 \theta + \delta_s a' \nabla^2 t, \qquad (7.47)$$

where $\delta_s = \dfrac{\delta}{c'_T}$ is the Soret coefficient for a moist body.

For a number of regimes of unsteady heat and mass transfer processes the Soret coefficient is an extremely small quantity, and the second term of the right-hand side of equation (7.47) can be neglected.

CHAPTER II

Non-Stationary Fields of Heat and Mass Transfer Potentials with Boundary Conditions of the Third Kind

Internal heat and mass transfer are described by a system of differential transfer equations (Chapter 1) which for uniform bodies can be expressed as

$$\frac{\partial t(\xi,\tau)}{\partial\tau}=a_q\left[\frac{\partial^2 t(\xi,\tau)}{\partial\xi^2}+\frac{\Gamma}{\xi}\,\frac{\partial t(\xi,\tau)}{\partial\xi}\right]+$$

$$+\frac{\varepsilon\rho c'_{\text{T}}}{c_q}\,\frac{\partial\theta(\xi,\tau)}{\partial\tau}, \tag{I'}$$

$$\frac{\partial\theta(\xi,\tau)}{\partial\tau}=a_m\left[\frac{\partial^2\theta(\xi,\tau)}{\partial\xi^2}+\frac{\Gamma}{\xi}\,\frac{\partial\theta(\xi,\tau)}{\partial\xi}\right]+$$

$$+a_m\delta\left[\frac{\partial^2 t(\xi,\tau)}{\partial\xi^2}+\frac{\Gamma}{\xi}\,\frac{\partial t(\xi,\tau)}{\partial\xi}\right], \tag{II'}$$

where

t — is the body temperature (heat transfer potential);

θ — the mass transfer potential

c_q and c'_{T} — the specific heat capacity and mass capacity, respectively;

$a_q=\frac{\lambda}{c\gamma_0}$ — thermal diffusion coefficient (coefficient of potential conductivity of heat transfer);

$a_m=\frac{\lambda_m}{c'_{\text{T}}\gamma_0}$ — coefficient of potential conductivity of mass transfer;

λ_m — mass transfer coefficient;

δ — Soret coefficient for a moist body;

ρ — specific heat of phase transformation;

Γ — a shape constant (for an infinite plate, $\Gamma=0$; for a cylinder, $\Gamma=1$; for a sphere, $\Gamma=2$);

ξ — a generalized coordinate selected according to the shape of the body (for an infinite plate and rod,

$\xi = x$, for an infinite cylinder and sphere, $\xi = r$),
ε — phase transformation number.

The system of differential equations (I'-II') can be written in a different form, if instead of the mass transfer potential, θ, we introduce the specific mass content (moisture content) of the body, u:

$$\frac{\partial t(\xi,\tau)}{\partial\tau} = a_q\left[\frac{\partial^2 t(\xi,\tau)}{\partial\xi^2} + \frac{\Gamma}{\xi}\,\frac{\partial t(\xi,\tau)}{\partial\xi}\right] + \frac{\varepsilon\rho}{c_q}\,\frac{\partial u(\xi,\tau)}{\partial\tau}, \tag{I}$$

$$\frac{\partial u(\xi,\tau)}{\partial\tau} = a_m\left[\frac{\partial^2 u(\xi,\tau)}{\partial\xi^2} + \frac{\Gamma}{\xi}\,\frac{\partial u(\xi,\tau)}{\partial\xi}\right] + a_m\,\delta\left[\frac{\partial^2 t(\xi,\tau)}{\partial\xi^2} + \frac{\Gamma}{\xi}\,\frac{\partial t(\xi,\tau)}{\partial\xi}\right]. \tag{II}$$

The system of equations (I-II) differs from the system of equations (I'-II') only that in the first case the coefficient δ denotes the Soret coefficient, and in the second case it denotes the thermal gradient coefficient. Furthermore, instead of the ratio of the mass capacity to heat capacity $\left(\frac{c'_T}{c}\right)$ the last term of the first equation contains instead the reciprocal of the heat capacity.

Therefore, the solutions obtained for the system of equations (I-II) may be utilized for the solution of the system of transfer equations (I'-II') by substituting for the ratio $\frac{c'_T}{c}$, the quantity $\left(\frac{1}{c_q}\right)$. It should also be noted that θ has to be substituted for the specific moisture content of the body, u. For the convenience of explanation we shall consider solutions of the system of equations (I-II) only. However, to underscore the close relationship which exists between these two sets of equations, the ratio $\frac{c'_T}{c_q}$ is denoted by $\frac{1}{c}$. Then, by substituting θ for u and the quantity c for c_q, we can get solutions for the system of equations (I'-II').

Analytical solutions for the system of equations (I-II) for boundary conditions of the first kind and for constant total specific moisture content were given by S.P. Kuznetsov and E.I. Sizyakova. The solutions for boundary conditions of the second kind with absence of thermal gradients resulting from mass transfer ($\delta = 0$), and for arbitrary initial conditions were obtained by M.S. Smirnov. The complete system of equations for internal heat and mass transfer for boundary conditions of the third kind and for the temperature of the medium represented by various functions of the time, were solved by Yu.A. Mikhaylov. A.V. Lykov solved the same equations for an infinite plate at constant t_c. Specific solutions of the incomplete system of equations (I–II) were given by F.M. Polonskaya, S.P. Kuznetsov, A.V. Lykov, P.D. Lebedev, Yu.A. Mikhaylov and M.S. Smirnov.

The effect of separate dimensionless criteria of similarity on heat and mass transfer processes has been investigated by A.V. Lykov, Yu.A. Mikhaylov, P.D. Lebedev and E.I. Sizyakova. On the basis of the analytic solutions, S.P. Kuznetsov and E.I. Sizyakova have developed rapid methods for the experimental determination of heat and mass transfer coefficients. A characteristic of the above solutions is that all of them were obtained by means of integral transform methods. Analysis and calculations show that the use of classical methods does not allow effective separation of the variables t and θ. The application of integral transform methods has been successfully employed to overcome the above difficulties.

The solutions cited below are for the system of equations for internal heat and mass transfer in classical uniform bodies.

The most general formulation of boundary conditions of the third kind for the system of differential equations (I–II) may be written as:

$$-\lambda \frac{\partial t(R,\tau)}{\partial \xi} + \alpha \left[t_c - t(R,\tau)\right] - (1-\varepsilon)\rho\beta\gamma_0\left[\theta(R,\tau) - \theta_p\right] = 0, \qquad \text{(III)}$$

$$a_m \frac{\partial \theta(R,\tau)}{\partial \xi} + a_m \delta \frac{\partial t(R,\tau)}{\partial \xi} + \beta\left[\theta(R,\tau) - \theta_p\right] = 0, \qquad \text{(IV)}$$

where α is the heat transfer coefficient, and β is the mass transfer coefficient; and

$$\frac{\partial t(0,\tau)}{\partial \xi} = \frac{\partial \theta(0,\tau)}{\partial \xi} = 0, \quad t(0,\tau) < \infty, \quad \theta(0,\tau) < \infty, \qquad \text{(V)}$$

$$t(\xi,0) = f(\xi); \quad \theta(\xi,0) = \varphi(\xi). \qquad \text{(VI)}$$

Of particular importance are the solutions of the sytem of equations (I,II) for general conditions (III) to (VI) with a uniform or parabolic initial distribution of heat and mass transfer potentials, i.e.,

$$f(\xi) = t_0 = \text{const}, \quad \varphi(\xi) = \theta_0 = \text{const},$$

$$f(\xi) = t_{\text{ц}} + \frac{\xi^2}{R^2}(t_n - t_{\text{ц}}), \quad \varphi(\xi) = \theta_{\text{ц}} - \frac{\xi^2}{R^2}(\theta_{\text{ц}} - \theta_n).$$

For example, the uniform initial distribution of transfer potentials can serve as an analogue of heating a moist body, the parabolic initial conditions can serve as the analogue of the period when the drying rate is falling, etc.

1. NON-STATIONARY FIELDS OF HEAT AND MASS POTENTIALS IN A CONSTANT TEMPERATURE MEDIUM

The general method of solving the complete system of equations for internal heat and mass transfer is illustrated by an example of finding the potential field of heat and mass transfer for bodies in the form of an infinite plate and sphere. The solutions given in this chapter have been obtained by means of Laplace transforms.

a) Infinite Plate

Constant initial conditions ($\Gamma = 0$, $\xi = x$, $f(x) = t_0 = \text{const}$, $\varphi(x) = \theta_0 = \text{const}$)
Applying the Laplace transform to equations (I–II) we get

$$st_L(x, s) - t_0 = a_q t''_L(x, s) + \frac{\varepsilon\rho}{c} s\theta_L(x, s) - \frac{\varepsilon\rho}{c}\theta_0, \tag{1.1}$$

$$s\theta_L(x, s) - \theta_0 = a_m \theta''_L(x, s) + a_m \delta t''_L(x, s). \tag{1.2}$$

The differential equation for the temperature, $t_L(x, s)$, is found by the following method: from equation (1.1) we solve for $\theta_L(x, s)$ determine its second derivative, $\theta''_L(x, s)$ and then substitute the resulting expressions into equation (1.2). Following simplification we get

$$t_L^{\text{IV}}(x, s) - \left(\frac{s}{a_q} + \frac{\varepsilon\rho\delta s}{a_q c} + \frac{s}{a_m}\right) t''_L(x, s) + \\ + \frac{s^2}{a_q a_m} t_L(x, s) - \frac{st_0}{a_q a_m} = 0. \tag{1.3}$$

The general integral of inhomogeneous ordinary differential equation

(1.3) can be determined using the Laplace transform again, this time with respect to x. The usual operations give

$$t_L(x, s) = \frac{t_0}{s} + C_1 e^{\sqrt{\frac{s}{a_q}}\nu_1 x} + C_2 e^{\sqrt{\frac{s}{a_q}}\nu_2 x} +$$

$$+ C_3 e^{-\sqrt{\frac{s}{a_q}}\nu_1 x} + C_4 e^{-\sqrt{\frac{s}{a_q}}\nu_2 x}, \tag{1.4}$$

where C_k $(k = 1, 2, 3, 4)$ are arbitrary constants:

$$\nu_j^2 = \frac{1}{2}\left[\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right) + \right.$$

$$\left. + (-1)^j \sqrt{\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}}\right], \ (j = 1,2). \tag{1.5}$$

The conditions of symmetry (V) are

$$t_L'(0, s) = \theta_L'(0, s) = 0.$$

Whence it follows that $C_1 = C_3$ and $C_2 = C_4$.

This means that the solution for temperature, as depicted in equation (1.4) can be rewritten as

$$t_L(x, s) = \frac{t_0}{s} + B_1 \mathrm{ch}\sqrt{\frac{s}{a_q}}\nu_1 x + B_2 \mathrm{ch}\sqrt{\frac{s}{a_q}}\nu_2 x. \tag{1.6}$$

here, B_1 and B_2 are new constants with respect to x.

The general solution for the transform of the mass transfer potential, $\theta_L(x, s)$ is obtained by substituting $t_L''(x, s)$, found from equation (1.6) and $t_L(x, s)$ into equation (1.1). We obtain:

$$\theta_L(x, s) = \frac{\theta_0}{s} + \frac{c}{\varepsilon\rho} B_1 (1 - \nu_1^2)\, \mathrm{ch}\sqrt{\frac{s}{a_q}}\nu_1 x +$$

$$+ \frac{c}{\varepsilon\rho} B_2 (1 - \nu_2^2)\, \mathrm{ch}\sqrt{\frac{s}{a_q}}\nu_2 x. \tag{1.7}$$

B_1 and B_2 are determined from boundary conditions (III) to (IV), which after Laplace transformation assume the following form:

$$\begin{aligned} -\lambda t'_L(R,s) + \alpha\left[\frac{t_c}{s} - t_L(R,s)\right] - {} \\ -(1-\varepsilon)\rho\beta\left[\theta_L(R,s) - \frac{\theta_p}{s}\right] = 0, \end{aligned} \tag{1.8}$$

$$\lambda_m \theta'_L(R,s) + \lambda_m \delta t'_L(R,s) + \beta\left[\theta_L(R,s) - \frac{\theta_p}{s}\right] = 0. \tag{1.9}$$

From equations (1.6) and (1.7) we find values of $t_L(R,s)$, $t'_L(R,s)$, $\theta_L(R,s)$, and $\theta'_L(R,s)$ and substitute these into boundary conditions (1.8) and (1.9). Then

$$\begin{aligned} -B_1\Bigg\{\lambda\sqrt{\frac{s}{a_q}}\nu_1 \operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_1 R + {} \\ + \left[\alpha + \frac{1-\varepsilon}{\varepsilon}c\beta(1-\nu_1^2)\right]\operatorname{ch}\sqrt{\frac{s}{a_q}}\nu_1 R\Bigg\} - {} \\ -B_2\Bigg\{\lambda\sqrt{\frac{s}{a_q}}\nu_2 \operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_2 R + \Bigg[\alpha + {} \\ + \frac{1-\varepsilon}{\varepsilon}c\beta(1-\nu_2^2)\Bigg]\operatorname{ch}\sqrt{\frac{s}{a_q}}\nu_2 R\Bigg\} + {} \\ + \frac{\alpha(t_c - t_0)}{s} - (1-\varepsilon)\rho\beta\frac{\theta_0 - \theta_p}{s} = 0, \end{aligned} \tag{1.10}$$

$$\begin{aligned} B_1\Bigg\{\left[\lambda_m \frac{c}{\varepsilon\rho}(1-\nu_1^2) + \lambda_m\delta\right]\sqrt{\frac{s}{a_q}}\nu_1 \operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_1 R + {} \\ + \beta\frac{c}{\varepsilon\rho}(1-\nu_1^2)\operatorname{ch}\sqrt{\frac{s}{a_q}}\nu_1 R\Bigg\} + {} \\ + B_2\Bigg\{\left[\lambda_m \frac{c}{\varepsilon\rho}(1-\nu_2^2) + \lambda_m\delta\right]\sqrt{\frac{s}{a_q}}\nu_2 \operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_2 R + {} \\ + \beta\frac{c}{\varepsilon\rho}(1-\nu_2^2)\operatorname{ch}\sqrt{\frac{s}{a_q}}\nu_2 R\Bigg\} + \beta\frac{\theta_0 - \theta_p}{s} = 0. \end{aligned} \tag{1.11}$$

Following simplification of equations (1.10) and (1.11), including the formation of certain dimensionless groups, we get a system of algebraic equations (1.12) for the determination of B1 and B2:

$$\left.\begin{aligned} &B_1Q_1 + B_2Q_2 - \frac{t_c - t_0}{s} + (1-\varepsilon)\frac{\rho\beta}{\alpha}\,\frac{\theta_0 - \theta_p}{s} = 0 \\ &B_1P_1 + B_2P_2 + \frac{\mathrm{Fe}}{\delta}\,\frac{\theta_0 - \theta_p}{s} = 0 \end{aligned}\right\}, \tag{1.12}$$

where

$$Q_j = [1 + (1 - \nu_j^2)K_1]\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\nu_j R +$$

$$+ \frac{1}{\mathrm{Bi}_q}\sqrt{\frac{s}{a_q}}\,\nu_j R\mathrm{sh}\sqrt{\frac{s}{a_q}}\nu_j R, \tag{1.13}$$

$$P_j = (1 - \nu_j^2)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\nu_j R +$$

$$+ [(1 - \nu_j^2) + \mathrm{Fe}]\frac{1}{\mathrm{Bi}_m}\sqrt{\frac{s}{a_q}}\nu_j R\mathrm{sh}\sqrt{\frac{s}{a_q}}\nu_j R. \tag{1.14}$$

The solution of the system of equations (1.12) yields

$$\begin{aligned} B_1 &= -\frac{(t_c - t_0)P_2 + \frac{\varepsilon\rho}{c}(\theta_0 - \theta_p)\ (Q_2 - K_1P_2)}{s\,(Q_2P_1 - Q_1P_2)}, \\ B_2 &= \frac{(t_c - t_0)P_1 + \frac{\varepsilon\rho}{c}(\theta_0 - \theta_p)(Q_1 - K_1P_1)}{s\,(Q_2P_1 - Q_1P_2)}. \end{aligned} \tag{1.15}$$

Consequently

$$t_L(x, s) - \frac{t_0}{s} = (t_o - t_0)\left\{\frac{P_1 + \varepsilon\,\mathrm{Ko}\,(Q_1 - K_1P_1)}{s\,(Q_2P_1 - Q_1P_2)}\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\nu_2 x - \right.$$

$$\left. - \frac{P_2 + \varepsilon\,\mathrm{Ko}(Q_2 - K_1P_2)}{s\,(Q_2P_1 - Q_1P_2)}\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\nu_1 x\right\}, \tag{1.16}$$

$$\frac{\theta_0}{s} - \theta_L(x, s) = (\theta_0 - \theta_p)\left\{\frac{Q_2 + \dfrac{1}{\varepsilon \mathrm{Ko}} P_2 - K_1 P_2}{s(Q_2 P_1 - Q_1 P_2)} \times\right.$$

$$\times (1 - \nu_1^2)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x - \frac{Q_1 + \dfrac{1}{\varepsilon \mathrm{Ko}} P_1 - K_1 P_1}{s(Q_2 P_1 - Q_1 P_2)} \times$$

$$\left.\times (1 - \nu_2^2)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x\right\}. \tag{1.17}$$

This is a general solution of the system of differential equations in terms of Laplace transforms (I-VI). To obtain the inverse of the transform solution we can utilize the general expansion theorem. The expansion theorem applies to expressions which can be represented in the form of a ratio of two generalized polynomials $\Phi_0(s)/\psi_0(s)$. Here it is assumed that the function $\psi_0(s)$ possesses only simple roots and that the degree of the polynomial in the denominator is larger by one than the degree of the polynomial in the numerator. The theorem can also be applied to the ratio of non-polynomials (non-generalized polynomials) $\Phi_1(s)$ and $\psi_1(s)$, when the latter by means of multiplication or division by s^l — ($|l| \leqslant 1$) can be reduced to the above form.

In our case each of the transform solutions can be represented as a ratio of two polynomials

$$\frac{1}{t_c - t_0}\left[t_L(x, s) - \frac{t_0}{s}\right] = \frac{\Phi_{01}(s)}{\psi_0(s)}, \tag{1.18}$$

$$\frac{1}{\theta_0 - \theta_p}\left[\frac{\theta_0}{s} - \theta_L(x, s)\right] = \frac{\Phi_{02}(s)}{\psi_0(s)} \tag{1.19}$$

The denominators, $\psi_0(s) = s(Q_2P_1 - Q_1P_2)$, in equations (1.18) and (1.19) are the same, and the polynomials in the numerators are equal to the following:

$$\Phi_{01}(s) = [P_1 + \varepsilon \mathrm{Ko}(Q_1 - K_1 P_1)]\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x -$$

$$- [P_2 + \varepsilon \mathrm{Ko}\,(Q_2 - K_1 P_2)]\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x,$$

$$\Phi_{02}(s) = \left(Q_2 + \frac{P_2}{\varepsilon \mathrm{Ko}} - K_1 P_2\right)(1 - \nu_1^2)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x -$$

$$-\left(Q_1 + \frac{P_1}{\varepsilon \mathrm{Ko}} - K_1 P_1\right)(1 - \nu_2^2)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x.$$

The numerator and denominator of equations (1.18) and (1.19) are general polynomials with respect to s; therefore, the inversion of the Laplace transform can be effected in accordance with the usual expansion theorem, that is,

$$L^{-1}\left[\frac{\Phi_{01}(s)}{\psi_0(s)}\right]=\frac{\Phi_{01}(0)}{\psi_0'(0)}+$$

$$+\sum_{n=1}^{\infty}\frac{\Phi_{01}(s_n)}{\psi_0'(s_n)}\exp(s_n\tau) \tag{1.18'}$$

$$L^{-1}\left[\frac{\Phi_{02}(s)}{\psi_0(s)}\right]=\frac{\Phi_{02}(0)}{\psi_0'(0)}+$$

$$+\sum_{n=1}^{\infty}\frac{\Phi_{02}(s_n)}{\psi_0'(s_n)}\exp(s_n\tau). \tag{1.19'}$$

Here, the symbol L^{-1} denotes the inverse Laplace transformation.

We now find the roots of $\psi_0(s)$, for which we assume that,

$$\psi_0(s)=s(Q_2P_1-Q_1P_2)=0.$$

Here we get

1) $s=s_0=0$ (zeroth root)

2) $s=s_n$ (where s_n satisifes the equation $Q_2P_1-Q_1P_2=0$).

Let us now define more precisely the values of the roots s_n. For this purpose we shall use the arguments of the hyperbolic sine and cosine in expressions (1.13) and (1.14) in accordance with the formulae shz $= -i\sin iz$ and chz $= \cos iz$. Here we get

$$s_n=-\frac{a_q\mu_n^2}{R^2},$$

where μ_n are the roots of the characteristic equation

$$Q_{n2}P_{n1}-Q_{n1}P_{n2}=0, \tag{1.20}$$

$$Q_{nj}=[1+(1-\nu_j^2)K_1]\cos\nu_j\mu_n-$$

$$-\frac{\nu_j\mu_n}{\mathrm{Bi}_q}\sin\nu_j\mu_n, \tag{1.21}$$

$$P_{nj}=(1-\nu_j^2)\cos\nu_j\mu_n-$$

$$-[(1-\nu_j^2)+\mathrm{Fe}]\frac{\nu_j\mu_n}{\mathrm{Bi}_m}\sin\nu_j\mu_n. \tag{1.22}$$

In order to obtain the inverse transforms (1.18') and (1.19'), we shall determine the quantities contained in them: $\psi_0'(0)$, $\psi_0'(s_n)$, $\Phi_{01}(0)$, $\Phi_{02}(0)$, $\Phi_{01}(s_n)$, $\Phi_{02}(s_n)$. Calculation shows

$$\psi_0' = \nu_2^2 - \nu_1^2,$$

$$\Phi_{01}(0) = \Phi_{02}(0) = \nu_2^2 - \nu_1^2,$$

$$\Phi_{01}(s_n) = -\Big\{[P_{n2} + \varepsilon\,\mathrm{Ko}\,(Q_{n2} - K_1 P_{n2})] \cos \nu_1 \mu_n \frac{x}{R} - [P_{n1} + \varepsilon\,\mathrm{Ko}(Q_{n1} - K_1 P_{n1})] \cos \nu_2 \mu_n \frac{x}{R}\Big\},$$

$$\Phi_{02}(s_n) = -\Big\{\Big(Q_{n1} + \frac{1}{\varepsilon\,\mathrm{Ko}} P_{n1} - K_1 P_{n1}\Big) \times (1 - \nu_2^2) \cos \nu_2 \mu_n \frac{x}{R} - \Big(Q_{n2} + \frac{1}{\varepsilon\,\mathrm{Ko}} P_{n2} - K_1 P_{n2}\Big) \times (1 - \nu_1^2) \cos \nu_1 \mu_n \frac{x}{R}\Big\},$$

$$\psi_0'(s_n) = \frac{1}{2} \mu_n \psi_n,$$

where

$$\psi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2}, \tag{1.23}$$

$$A_{nj} = \Big[1 + \frac{1}{Bi_q} + (1 - \nu_j^2) K_1\Big] \sin \nu_j \mu_n + \frac{\nu_j \mu_n}{Bi_q} \cos \nu_j \mu_n, \tag{1.24}$$

$$B_{nj} = (1 - \nu_j^2) \sin \nu_j \mu_n + \frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m} (\sin \nu_j \mu_n + \nu_j \mu_n \cos \nu_j \mu_n). \tag{1.25}$$

We can now substitute the quantities found above into equations (1.18') and (1.19'); as a result, for the unsteady fields of heat and mass transfer potentials we get

$$\frac{t(x,\tau)-t_0}{t_c-t_0}=1-2\sum_{n=1}^{\infty}\left\{\frac{P_{n2}+\varepsilon\,\mathrm{Ko}(Q_{n2}-K_1P_{n2})}{\mu_n\,\psi_n}\times\right.$$

$$\times\cos\nu_1\mu_n\frac{x}{R}-\frac{P_{n1}+\varepsilon\,\mathrm{Ko}(Q_{n1}-K_1P_{n1})}{\mu_n\,\psi_n}\times$$

$$\left.\times\cos\nu_2\mu_n\frac{x}{R}\right\}\exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (1.26)$$

$$\frac{\theta_0-\theta(x,\tau)}{\theta_0-\theta_p}=1-2\sum_{n=1}^{\infty}\left\{\frac{Q_{n1}+\dfrac{1}{\varepsilon\,\mathrm{Ko}}P_{n1}-K_1P_{n1}}{\mu_n\,\psi_n}\times\right.$$

$$\times(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R}-\frac{Q_{n2}+\dfrac{1}{\varepsilon\,\mathrm{Ko}}P_{n2}-K_1P_{n2}}{\mu_n\,\psi_n}\times$$

$$\left.\times(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R}\right\}\exp\left(-\mu_n^2\,\mathrm{Fo}\right). \qquad (1.27)$$

Thus, the system of differential equations of internal heat and mass transfer (I)-(VI) for an infinite plate with boundary conditions of the third kind and a uniform initial distribution of the transfer potentials has been solved. In order to utilize the solution we can rewrite it in a more convenient form by combining equations (1.5) and (1.20) to (1.27):

$$T=\frac{t(x,\tau)-t_0}{t_c-t_0}=1-\sum_{n=1}^{\infty}\left(C_{n2}\cos\nu_1\mu_n\frac{x}{R}-\right.$$

$$\left.-C_{n1}\cos\nu_2\mu_n\frac{x}{R}\right)\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (1.28)$$

$$\Theta=\frac{\theta_0-\theta(x,\tau)}{\theta_0-\theta_p}=1-\sum_{n=1}^{\infty}\left[C_{n1}^{*}(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R}\right.$$

$$\left.-C_{n2}^{*}(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R}\right]\exp(-\mu_n^2\,\mathrm{Fo}). \qquad (1.29)$$

here, ν_j are coefficients $(j=1;\ 2)$ determined by

$$\nu_j^2 = \frac{1}{2}\left[\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right) + \right.$$

$$\left. + (-1)^j \sqrt{\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}}\,\right],$$

C_{nj}, C^*_{nj} are determined by

$$C_{nj} = 2\,\frac{P_{nj} + \varepsilon\,\mathrm{Ko}(Q_{nj} - K_1 P_{nj})}{\mu_n \psi_n},$$

$$C^*_{nj} = \frac{C_{nj}}{\varepsilon\,\mathrm{Ko}}, \tag{1.30}$$

$$\psi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2},$$

$$A_{nj} = \left[1 + \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2) K_1\right] \sin \nu_j \mu_n +$$

$$+ \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n,$$

$$B_{nj} = (1 - \nu_j^2) \sin \nu_j \mu_n +$$

$$+ \frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m} (\sin \nu_j \mu_n + \nu_j \mu_n \cos \nu_j \mu_n),$$

$$Q_{nj} = [1 + (1 - \nu_j^2) K_1] \cos \nu_j \mu_n -$$

$$- \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \sin \nu_j \mu_n,$$

$$P_{nj} = (1 - \nu_j^2) \cos \nu_j \mu_n -$$

$$- [(1 - \nu_j^2) + \mathrm{Fe}] \frac{\nu_j \mu_n}{\mathrm{Bi}_m} \sin \nu_j \mu_n.$$

μ_n are the roots of the characteristic equation:

$$Q_{n2} P_{n1} = Q_{n1} P_{n2}. \tag{1.20}$$

The values of the characteristic roots, μ_n, can be obtained by a graphical method. For this purpose we can rewrite equation (1.20) in a somewhat different form:

$$\frac{M}{N} = \frac{\mu}{\mathrm{Bi}_q}, \tag{1.31}$$

where

$$M = P_1 \cos \nu_2 \mu - P_2 \cos \nu_1 \mu,$$

$$N = \left[(1-\nu_1^2) \frac{1-\varepsilon}{\varepsilon} \mathrm{Lu\,Bi}_m \frac{\cos \nu_1 \mu}{\mu} - \nu_1 \sin \nu_1 \mu \right] P_2 - \left[(1-\nu_2^2) \frac{1-\varepsilon}{\varepsilon} \mathrm{Lu\,Bi}_m \frac{\cos \nu_2 \mu}{\mu} - \nu_2 \sin \nu_2 \mu \right] P_1.$$

If the left part of equation (1.31) is denoted by $\Phi_\mu = \Phi(\mu)$, and the right part by $y = \frac{1}{\mathrm{Bi}_q} \mu$, then the point of intersection of curves Φ_μ with the straight line, y, on the graph of $[\mu, \Phi_\mu]$ will yield the value of roots μ_n, at any given combination of the following dimensionless factors: Lu, ε, Ko, Pn, Bi_m, Bi_q (Fig. 1). For various values of the Bi_q number (from 0.10 to infinity) as a function of Lu, Bi_m, Fe, and ε, the characteristic roots μ_n are collected and presented in Tables 1-4. For practical convenience a graphical representation of μ_n for the indicated dimensionless numbers is given in Figures 2-9.

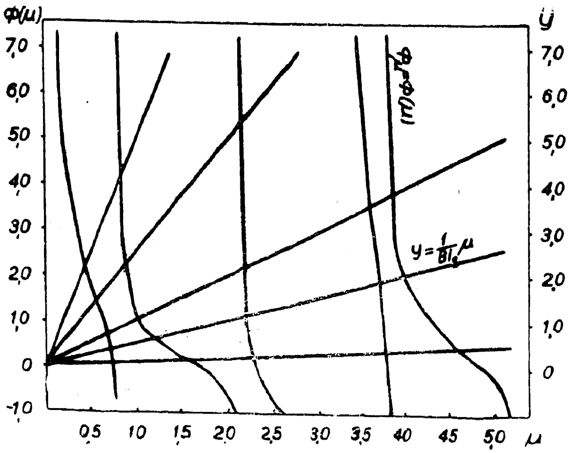

Fig. 1. Method of determining roots of the characteristic equation $\Phi(\mu) = \frac{1}{\mathrm{Bi}_q}\mu$ for an infinite plate.

Examination of equation (1.31) and Figure 1 shows that we have an infinite number of roots μ_n whose values increase in the following manner:

$$\mu_1 < \mu_2 < \ldots \mu_n < \ldots$$

As a result of this inequality, the infinite series in equations (1.28) and (1.29) converge quite rapidly. It may be seen from Table 5 and Figure 10 that the percent error introduced by neglecting terms of the series after the first two which appear under the summation sign diminishes with increasing dimensionless time (F0).

For a given value of the Fourier number a certain degree of accuracy can be achieved with only the first two terms of the entire series. The starting values of the Fourier number, beyond which the error is less than 1.0 or 0.5%, are presented in Table 6. These values can be used for simplified calculations.

Simplifications are of considerable importance because they lead to tremendous reduction in the volume of computation and to the possibility of presenting general solutions in a form convenient for practical applications. The starting values of the Fourier numbers for simplified calculations change as the values of the parameters, included in the solution, change.

Analysis of the calculations shows that the largest influence on the simplification (on the magnitude of the starting Fo) is indicated by values of the Lu numbers. As Lu increases the magnitude of the starting Fo decreases. The influence of other dimensionless numbers upon the simplification is considerably smaller. Table 6 shows that the starting values of Fo as a function of Bi_q, increase with increasing Bi_q for local values of temperature and decreases for average values of temperature. This influence of the Biot number has an especially significant effect on the dimensionless heat potential. Analogously, the starting Fo value for the dimensionless mass potential is especially dependent on the Bi_m number. Generally speaking, the value of the starting Fo is lower for heat transfer than for mass transfer.

It can therefore be concluded that the dimensionless transfer potentials in an infinite plate are functions of the dimensionless coordinate, time, and of the following parameters: Lu, Bi_q, Bi_m, ε, Ko and Pn, i.e.:

$$T = T(x/R,\ \mathrm{Fo},\ \mathrm{Lu},\ \mathrm{Bi}_q,\ \mathrm{Bi}_m,\ \varepsilon,\ \mathrm{Ko},\ \mathrm{Pn}),$$

$$\Theta = \Theta(x/R,\ \mathrm{Fo},\ \mathrm{Lu},\ \mathrm{Bi}_q,\ \mathrm{Bi}_m,\ \varepsilon,\ \mathrm{Ko},\ \mathrm{Pn}),$$

because

$$C_{nj} = \psi_1(\mathrm{Lu},\ \mathrm{Bi}_q,\ \mathrm{Bi}_m,\ \varepsilon,\ \mathrm{Ko},\ \mathrm{Pn}),$$

$$C^*_{nj} = \psi_2(\mathrm{Lu},\ \mathrm{Bi}_q,\ \mathrm{Bi}_m,\ \varepsilon,\ \mathrm{Ko},\ \mathrm{Pn})$$

The numerical values of the first two coefficients C_{ni} and C^*_{nj} as a function of the dimensionless numbers cited are given in Tables 7 to 12, and for the simplification of calculations are given in Figures 11 to 16.

Table 1. Roots of the characteristic equation $\Phi(\mu) = \frac{1}{\mathrm{Bi}_q}\mu$ as a function of Lu (infinite plate) at: $\mathrm{Fe}=\varepsilon$, $\mathrm{KoPn}=0,3$, $\varepsilon=0,5$ $\mathrm{Bi}_m=10$

Bi_q	Lu=0,15		Lu=0,3					Lu=0,5			Lu=0,7		Lu=1,0		Lu=2,0	
	μ_1	μ_2	μ_1	μ_2	μ_3	μ_4	μ_5	μ_1	μ_2	μ_3	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0,10	0,306	0,569	0,307	0,814				0,307			0,307		0,308		0,308	
0,15	0,367	0,576	0,370	0,818	2,168			0,371	1,077	2,508	0,372	1,307	0,373		0,373	
0,20	0,415	0,583	0,423	0,825	2,169	3,365	3,809	0,424	1,084	2,511	0,425	1,314	0,426	1,637	0,424	
0,25	0,451	0,596	0,465	0,832	2,170	3,379	3,810	0,467	1,090	2,515	0,468	1,318	0,469	1,643	0,468	
0,30	0,477	0,613	0,500	0,839	2,171	3,396	3,811	0,505	1,096	2,519	0,506	1,322	0,507	1,649	0,507	
0,35	0,494	0,635	0,530	0,847	2,172	3,408	3,812	0,537	1,101	2,523	0,540	1,329	0,542	1,655	0,543	
0,40	0,506	0,658	0,556	0,856	2,173	3,423	3,814	0,568	1,104	2,527	0,571	1,335	0,573	1,660	0,574	
0,45	0,513	0,682	0,579	0,867	2,174	3,434	3,816	0,593	1,107	2,534	0,596	1,342	0,599	1,666	0,602	
0,50	0,518	0,706	0,599	0,877	2,175	3,444	3,818	0,616	1,112	2,537	0,620	1,347	0,623	1,674	0,627	
0,55	0,522	0,730	0,616	0,888	2,176	3,454	3,820	0,637	1,120	2,542	0,642	1,358	0,646	1,677	0,649	
0,60	0,525	0,752	0,630	0,898	2,178	3,468	3,822	0,654	1,128	2,546	0,661	1,359	0,666	1,681	0,669	
0,70	0,529	0,796	0,652	0,925	2,180	3,491	3,825	0,688	1,147	2,552	0,696	1,372	0,702	1,689	0,709	
0,80	0,532	0,834	0,668	0,950	2,183	3,515	3,830	0,716	1,167	2,557	0,727	1,384	0,734	1,701	0,745	
0,90	0,533	0,869	0,681	0,973	2,187	3,537	3,836	0,738	1,183	2,564	0,753	1,396	0,762	1,710	0,774	
1,0	0,535	0,903	0,690	0,999	2,191	3,558	3,841	0,757	1,198	2,572	0,775	1,407	0,786	1,717	0,799	
1,5	0,538	1,028	0,716	1,106	2,195	3,639	3,878	0,817	1,269	2,600	0,857	1,459	0,878	1,753	0,900	

Table 1, continued

Bi_q	Lu=0,15		Lu=0,3					Lu=0,5			Lu=0,7		Lu=1,0		Lu=2,0	
	μ_1	μ_2	μ_1	μ_2	μ_3	μ_4	μ_5	μ_1	μ_2	μ_3	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
2,0	0,539	1,114	0,726	1,188	2,197	3,685	3,936	0,848	1,329	2,625	0,901	1,503	0,937	1,780	0,967	
3,0	0,539	1,238	0,735	1,298	2,204	3,725	4,074	0,877	1,412	2,657	0,951	1,566	0,996	1,817	1,047	
4,0	0,540	1,311	0,739	1,364	2,213	3,742	4,187	0,891	1,469	2,682	0,974	1,606	1,033	1,841	1,091	2,760
5,0	0,540	1,362	0,741	1,409	2,220	3,749	4,281	0,900	1,509	2,698	0,989	1,637	1,057	1,858	1,125	2,666
6,0	0,540	1,398	0,742	1,440	2,224	3,753	4,357	0,906	1,541	2,711	0,999	1,659	1,072	1,871	1,148	2,623
7,0	0,541	1,425	0,743	1,465	2,229	3,756	4,420	0,907	1,561	2,720	1,007	1,675	1,083	1,879	1,163	2,595
8,0	0,541	1,444	0,743	1,487	2,231	3,759	4,479	0,912	1,579	2,728	1,011	1,689	1,092	1,887	1,178	2,576
9,0	0,541	1,459	0,744	1,504	2,233	3,760	4,518	0,914	1,595	2,735	1,015	1,700	1,099	1,893	1,188	2,561
10	0,541	1,472	0,744	1,516	2,235	3,762	4,554	0,915	1,606	2,740	1,017	1,709	1,106	1,898	1,197	2,552
15	0,542	1,509	0,745	1,552	2,241	3,764	4,665	0.919	1,638	2,756	1,027	1,734	1,122	1,912	1,224	2,526
20	0,542	1,530	0,746	1,572	2,245	3,766	4,730	0,922	1,653	2,766	1,033	1,748	1,128	1,919	1,237	2,512
30	0,542	1,550	0,747	1,597	2,248	3,767	4,589	0,924	1,675	2,775	1,036	1,764	1,138	1,928	1,253	2,498
40	0,542	1,558	0,747	1,607	2,251	3,768	4,826	0,925	1,684	2,782	1,039	1,774	1,143	1,933	1,261	2,490
50	0,542	1,565	0,747	1,614	2,253	3,769	4,840	0,926	1,691	2,786	1,040	1,780	1,146	1,936	1,266	2,485
∞	0,542	1,580	0,748	1,638	2,256	3,770	4,911	0,929	1,711	2,801	1,045	1,797	1,155	1,943	1,284	2,469

Table 2. Roots of the characteristic equation $\Phi(\mu) = \frac{1}{\mathrm{Bi}_q}\mu$ as a function of $\mathrm{Bi_m}$ (infinite plate) for $\mathrm{Lu}=0{,}3$, $\mathrm{KoPn}=0{,}6$, $\varepsilon=0{,}5$*

Bi_q	$\mathrm{Bi}_m=0{,}1$		$\mathrm{Bi}_m=0{,}5$		$\mathrm{Bi}_m=5$		$\mathrm{Bi}_m=7$		$\mathrm{Bi}_m=15$		$\mathrm{Bi}_m=20$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0,10	0,313	1,640	0,302	0,370	0,307	0,746	0,307	0,783	0,307	0,839	0,307	0,854
0,15	0,379	1,641	0,339	0,401	0,370	0,752	0,370	0,786	0,370	0,844	0,370	0,859
0,20	0,434	1,642	0,348	0,449	0,419	0,759	0,422	0,793	0,423	0,850	0,423	0,865
0,25	0,482	1,643	0,350	0,493	0,460	0,767	0,463	0,800	0,465	0,856	0,466	0,870
0,30	0,524	1,644	0,352	0,534	0,497	0,774	0,499	0,809	0,501	0,863	0,502	0,876
0,35	0,562	1,646	0,352	0,571	0,526	0,783	0,529	0,816	0,531	0,871	0,533	0,884
0,40	0,595	1,647	0,352	0,606	0,550	0,793	0,554	0,826	0,557	0,878	0,559	0,892
0,45	0,626	1,648	0,353	0,637	0,570	0,805	0,577	0,838	0,581	0,888	0,583	0,902
0,50	0,653	1,649	0,353	0,664	0,586	0,818	0,595	0,850	0,601	0,898	0,603	0,912
0,55	0,680	1,650	0,353	0,691	0,602	0,832	0,610	0,863	0,618	0,908	0,622	0,923
0,60	0,705	1,652	0,353	0,716	0,615	0,847	0,623	0,875	0,633	0,918	0,637	0,934
0,70	0,751	1,655	0,354	0,762	0,632	0,877	0,643	0,900	0,657	0,941	0,661	0,955
0,80	0,790	1,657	0,354	0,803	0,643	0,906	0,658	0,927	0,676	0,965	0,680	0,975
0,90	0,826	1,659	0,354	0,839	0,651	0,937	0,668	0,954	0,690	0,990	0,694	0,998
1,0	0,859	1,660	0,354	0,872	0,657	0,965	0,676	0,981	0,701	1,014	0,706	1,020

Table 2, continued

Bi_q	$Bi_m=0{,}1$		$Bi_m=0{,}5$		$Bi_m=5$		$Bi_m=7$		$Bi_m=15$		$Bi_m=20$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
1,5	0,984	1,671	0,354	0,997	0,672	1,031	0,695	1,094	0,730	1,117	0,737	1,121
2,0	1,070	1,682	0,355	1,088	0,679	1,166	0,704	1,117	0,742	1,197	0,750	1,200
3,0	1,180	1,701	0,355	1,198	0,684	1,278	0,711	1,285	0,752	1,302	0,760	1,308
4,0	1,245	1,718	0,355	1,260	0,686	1,348	0,714	1,358	0,756	1,373	0,765	1,379
5,0	1,286	1,730	0,355	1,304	0,687	1,393	0,716	1,403	0,759	1,420	0,768	1,425
6,0	1,316	1,740	0,355	1,337	0,687	1,426	0,717	1,436	0,760	1,450	0,770	1,455
7,0	1,336	1,751	0,355	1,359	0,688	1,451	0,718	1,462	0,761	1,476	0,771	1,481
8,0	1,352	1,761	0,355	1,377	0,688	1,473	0,718	1,482	0,762	1,499	0,772	1,504
9,0	1,363	1,767	0,355	1,388	0,689	1,487	0,719	1,496	0,762	1,513	0,773	1,519
10	1,370	1,773	0,355	1,396	0,689	1,499	0,720	1,508	0,763	1,526	0,774	1,531
15	1,397	1,795	0,355	1,428	0,690	1,537	0,721	1,546	0,765	1,564	0,775	1,570
20	1,409	1,807	0,355	1,444	0,690	1,557	0,721	1,566	0,765	1,585	0,776	1,589
30	1,421	1,822	0,355	1,458	0,690	1,579	0,722	1,586	0,766	1,606	0,777	1,611
40	1,427	1,829	0,355	1,465	0,690	1,589	0,722	1,596	0,766	1,616	0,777	1,621
50	1,430	1,835	0,355	1,470	0,690	1,595	0,722	1,603	0,767	1,622	0,778	1,627
∞	1,443	1,861	0,355	1,485	0,691	1,621	0,723	1,628	0,767	1,650	0,778	1,652

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* For $Bi_m = 10$, see Table 1

Table 3. Roots of the characteristic equation $\Phi(\mu) = \frac{1}{\mathrm{Bi}_q}\mu$ as a function of Fe = ε KoPn (infinite plate) for $\mathrm{Lu} = 0{,}3$, $\varepsilon = 0{,}5$, $\mathrm{Bi}_m = 10^{8}$

Bi_q	Fe = 0,1		Fe = 0,2		Fe = 0,4		Fe = 0,5		Fe = 0,6	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_i	μ_2
0,10	0,310	0,793	0,309	0,803	0,306	0,824	0,305	0,835	0,304	0,845
0,15	0,375	0,794	0,373	0,807	0,368	0,832	0,366	0,845	0,364	0,858
0,20	0,428	0,796	0,426	0,811	0,419	0,840	0,415	0,853	0,412	0,870
0,25	0,474	0,800	0,469	0,816	0,461	0,848	0,455	0,862	0,451	0,881
0,30	0,513	0,803	0,507	0,820	0,494	0,855	0,488	0,874	0,482	0,895
0,35	0,548	0,806	0,539	0,826	0,523	0,866	0,516	0,887	0,510	0,908
0,40	0,579	0,809	0,567	0,833	0,547	0,877	0,541	0,899	0,534	0,925
0,45	0,606	0,813	0,592	0,841	0,569	0,889	0,562	0,913	0,554	0,939
0,50	0,630	0,818	0,613	0,849	0,587	0,902	0,578	0,927	0,570	0,954
0,55	0,650	0,825	0,632	0,858	0,604	0,914	0,593	0,940	0,583	0,968
0,60	0,669	0,834	0,647	0,869	0,617	0,928	0,605	0,954	0,594	0,982
0,70	0,696	0,852	0,671	0,891	0,633	0,954	0,626	0,982	0,614	1,009
0,80	0,715	0,873	0,689	0,916	0,654	0.980	0,641	1,012	0,629	1,036
0,90	0,728	0,897	0,702	0,940	0,666	1,007	0,653	1,039	0,641	1,065
1,0	0,737	0,922	0,711	0,964	0,676	1,032	0,662	1,064	0,650	1,091

Table 3, continued

Bi_q	Fe = 0,1		Fe = 0,2		Fe = 0,4		Fe = 0,5		Fe = 0,6	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
1,5	0,753	1,033	0,733	1,071	0,701	1,139	0,687	1,171	0,676	1,205
2,0	0,760	1,119	0,742	1,153	0,712	1,219	0,699	1,251	0,688	1,286
3,0	0,765	1,227	0,748	1,264	0,721	1,326	0,710	1,360	0,699	1,390
4,0	0,768	1,301	0,752	1,333	0,726	1,393	0,714	1,427	0,704	1,457
5,0	0,769	1,351	0,753	1,380	0,728	1,441	0,717	1,472	0,707	1,501
6,0	0,770	1,383	0,754	1,409	0,729	1,470	0,719	1,501	0,709	1,531
7,0	0,770	1,410	0,755	1,438	0,730	1,496	0,720	1,524	0,710	1,553
8,0	0,770	1,431	0,756	1,459	0,731	1,577	0,721	1,543	0,711	1,571
9,0	0,770	1,443	0,756	1,475	0,732	1,581	0,722	1,559	0,712	1,584
10	0,770	1,459	0,757	1,485	0,733	1,541	0,723	1,567	0,713	1,595
15	0,770	1,504	0,758	1,530	0,734	1,584	0,725	1,607	0,714	1,631
20	0,770	1,527	0,758	1,551	0,735	1,602	0,726	1,624	0,716	1,648
30	0,771	1,554	0,758	1,574	0,735	1,625	0,726	1,644	0,716	1,666
40	0,771	1,564	0,759	1,584	0,736	1,633	0,727	1,653	0,717	1,676
50	0,771	1,574	0,759	1,593	0,736	1,641	0,727	1,661	0,717	1,683
∞	0,771	1,603	0,759	1,618	0,737	1,665	0,728	1,683	0,718	1,704

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* Fe = 0.3, see Table 1

Table 4. Roots of the Characteristic equation $\Phi(\mu) = \frac{1}{\mathrm{Bi}_q}\mu$ as a function of ε (infinite plate) for $\mathrm{Lu} = 0{,}3$, $\mathrm{KoPn} = 0{,}6$, $\mathrm{Bi}_m = 10$*

Bi_q	$\varepsilon = 0{,}1667$		$\varepsilon = 0{,}3333$		$\varepsilon = 0{,}6667$		$\varepsilon = 0{,}8333$		$\varepsilon = 1{,}000$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0,10	0,307	0,836	0,307	0,825	0,307	0,803	0,307	0,793	0,307	0,783
0,15	0,373	0,840	0,372	0,830	0,371	0,809	0,371	0,798	0,370	0,789
0,20	0,423	0,846	0,423	0,835	0,422	0,815	0,422	0,806	0,421	0,797
0,25	0,465	0,852	0,465	0,841	0,464	0,823	0,464	0,813	0,463	0,802
0,30	0,501	0,858	0,500	0,849	0,499	0,831	0,498	0,819	0.497	0,812
0,35	0,533	0,866	0,532	0,855	0,529	0,839	0,528	0,830	0,526	0,821
0,40	0,559	0,872	0,558	0,863	0,555	0,848	0,553	0,840	0,551	0,832
0,45	0,583	0,881	0,581	0,873	0,577	0,859	0,575	0,852	0,572	0,845
0,50	0,603	0,890	0,601	0,883	0,595	0,870	0,593	0,863	0,591	0,857
0,55	0,622	0,900	0,619	0,894	0,614	0,882	0,611	0,876	0,607	0,871
0,60	0,638	0,910	0,634	0,905	0,628	0,894	0,625	0,888	0,620	0,885
0,70	0,662	0,931	0,658	0,927	0,650	0,919	0,645	0,915	0,639	0,912
0,80	0,681	0,952	0,676	0,951	0,664	0,945	0,658	0,942	0,653	0,941
0,90	0,695	0,975	0,689	0,974	0,675	0,973	0,670	0,971	0,664	0,970
1,0	0,705	0,999	0,698	0,999	0,683	0,999	0,677	0,999	0,671	0,999

Table 4, continued

Bi_q	$\varepsilon = 0,1667$		$\varepsilon = 0,3333$		$\varepsilon = 0,6667$		$\varepsilon = 0,8333$		$\varepsilon = 1,000$	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
1,5	0,733	1,096	0,723	1,101	0,706	1,109	0,698	1,112	0,692	1,115
2,0	0,745	1,172	0,733	1,179	0,716	1,190	0,707	1,197	0,700	1,203
3,0	0,755	1,273	0,743	1,285	0,725	1,304	0,715	1,314	0,707	1,322
4,0	0,759	1,340	0,747	1,350	0,728	1,379	0,718	1,387	0,710	1,400
5,0	0,761	1,382	0,749	1,393	0,730	1,425	0,720	1,436	0,712	1,450
6,0	0,762	1,414	0,751	1,427	0,732	1,459	0,721	1,475	0,713	1,487
7,0	0,763	1,435	0,752	1,450	0,733	1,484	0,722	1,499	0,714	1,511
8,0	0,764	1,454	0,753	1,470	0,734	1,505	0,723	1,520	0,715	1,534
9,0	0,765	1,467	0,754	1,486	0,734	1,521	0,723	1,537	0,715	1,552
10	0,766	1,477	0,754	1,499	0,735	1,536	0,724	1,553	0,716	1,566
15	0,767	1,513	0,756	1,532	0,736	1,572	0,725	1,590	0,717	1,606
20	0,767	1,533	0,757	1,553	0,736	1,593	0,725	1,612	0,717	1,629
30	0,768	1,554	0,757	1,573	0,737	1,615	0,726	1,634	0,717	1,653
40	0,768	1,564	0,758	1,583	0,737	1,626	0,726	1,645	0,718	1,664
50	0,769	1,575	0,758	1,593	0,738	1,637	0,727	1,657	0,718	1,676
∞	0,770	1,605	0,758	1,617	0,738	1,666	0,727	1,684	0,719	1,707

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* For $\varepsilon = 0.5$, see Table 1

Table 5

Fo	% of error			
	$1-T$		$1-\Theta$	
	$x/R = 0$	$x/R = 1$	$x/R = 0$	$x/R = 1$
0,1	11,30	40,16	18,05	137,37
0,2	6,62	20,53	18,72	71,46
0,3	4,20	12,74	9,23	40,05
0,4	3,21	8,48	6,00	23,48
0,5	2,26	5,75	3,85	14,15
0,6	1,59	3,91	2,47	8,68
0,7	1,10	2,65	1,58	5,37
0,8	0,76	1,79	1,01	3,35
0,9	0,52	1,20	0,64	2,10
1,0	0,35	0,80	0,41	1,32

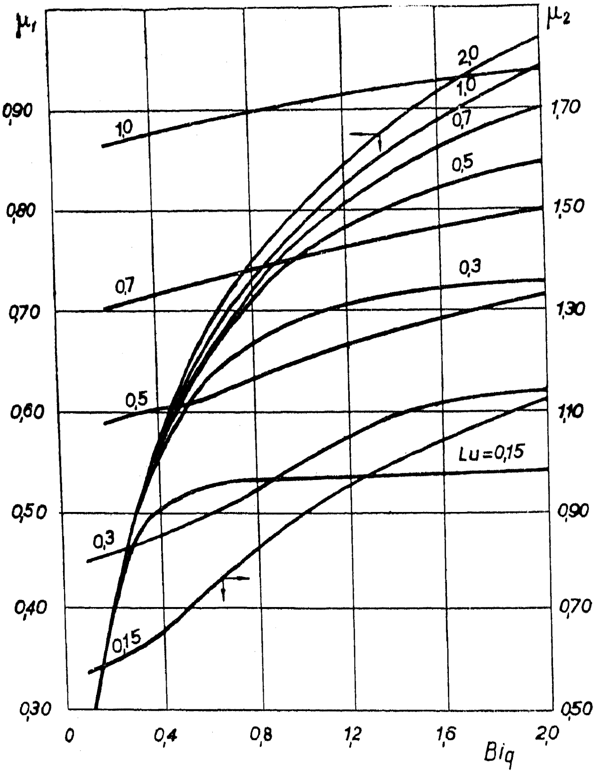

Fig. 2. Characteristic roots as a function of Lu for an infinite plate ($Bi_q = 0{,}1 \div 2{,}0$).

Table 6. Values of starting Fo for simplified calculations of heat and mass transfer.

($\mathrm{Lu} = 0{,}3$, $\mathrm{Ko} = 1{,}2$, $\mathrm{Pn} = 0{,}5$, $\varepsilon = 0{,}5$, $\mathrm{Bi}_m = 10$)

Type of Problem	Bi_q	Fo								Error
		$1 - T$		$1 - \theta$		$1-\overline{T}$	$1-\overline{\theta}$	$\frac{d\overline{T}}{d\mathrm{Fo}}$	$\frac{d\overline{\theta}}{d\mathrm{Fo}}$	$\lvert\Delta\rvert$ %
		$x/R = 0$	$x/R = 1$	$x/R = 0$	$x/R = 1$					
Plate, constant initial conditions	1	0,75	0,66	0,79	~ 1,06	0,48	0,59	1,03	~ 1,13	1
		0,79	0,81	0,94	~ 1,19	0,61	0,74	1,19	~ 1,31	0,5
	10	0,73	0,95	0,80	1,06	0,11	0,57	0,29	1,06	1
		0,89	1,08	0,95	1,19	0,17	0,73	0,37	~ 1,19	0,5
Sphere, constant initial conditions	10		0,24		0,27	0,17	0,13	0,26	0,22	1
			0,27		0,31	0,21	0,15	0,30		0,5
Plate, parabolic initial conditions	10	0,66	0,87	0,73	1,00	0,13	0,50	0,25	0,33	1
		0,84	1,04	0,88	1,11	0,16	0,64	0,33	1,15	0,5

In order to study the processes of internal heat and mass transfer it is convenient to know the average temperature and mass potential. Also, the local and average velocity and their rate of change ought to be known. For example, the average temperature allows the calculation of the total heat transferred for the heating of a unit volume according to the following equation:

$$\Delta Q = c\gamma_0 [\bar{t}(\tau) - t_0]$$

Knowledge of the functional forms of the average and local rates makes it possible to obtain the most important characteristics of a number of processes (for drying, for example, the temperature and drying rate curves).

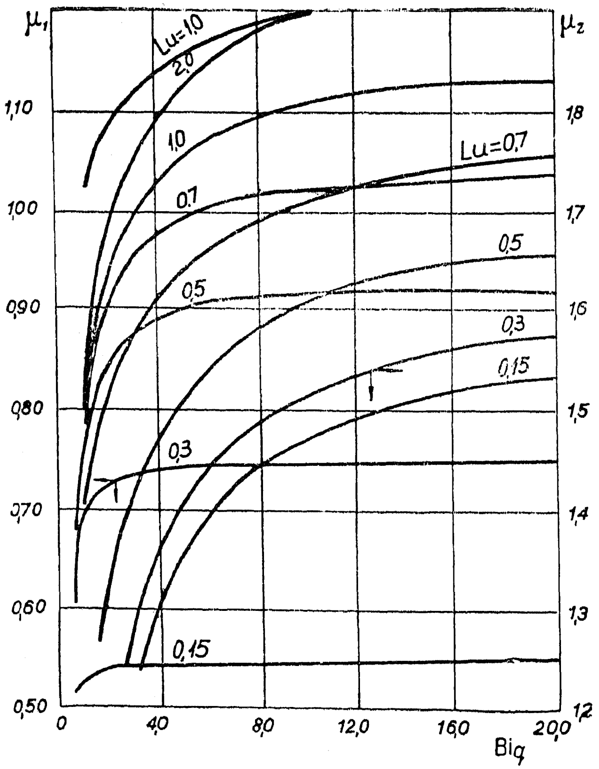

Fig. 3. Characteristic roots as a function of Lu for an infinite plate ($Bi_q = 1 \div 20$).

The average values of transfer potentials in an infinite plate can be obtained from the following formulas:

$$\bar{t}(\tau) = \frac{1}{R}\int_0^R t(x, \tau)\, dx, \quad \bar{\Theta}(\tau) = \frac{1}{R}\int_0^R \Theta(x, \tau)\, dx.$$

If the appropriate expressions from equations (1.28) and (1.29) are substituted for $t(x, \tau)$ and $\Theta(x, \tau)$, then, after integrating, the following expression is obtained for the average values of dimensionless temperatures and mass potentials:

$$T = \frac{\bar{t}(\tau) - t_0}{t_c - t_0} = 1 - \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.36}$$

$$\Theta = \frac{\theta_0 - \bar{\theta}(\tau)}{\theta_0 - \theta_p} = 1 - \sum_{n=1}^{\infty} D_n^* \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.37}$$

where

$$D_n = D_{n2} - D_{n1}, \quad D_n^* = D_{n1}^* - D_{n2}^*, \qquad (1.38$$

$$D_{n1} = \frac{\sin \nu_2 \mu_n}{\nu_2 \mu_n} C_{n1}, \quad D_{n2} = \frac{\sin \nu_1 \mu_n}{\nu_1 \mu_n} C_{n2},$$

$$D_{n1}^* = \frac{\sin \nu_2 \mu_n}{\nu_2 \mu_n}(1 - \nu_2^2) C_{n1}^*, \quad D_{n2}^* = \frac{\sin \nu_1 \mu_n}{\nu_1 \mu_n}(1 - \nu_1^2) C_{n2}^*. \tag{1.39}$$

The calculated values of the coefficients D_n and D_n^* and their graphical representations as functions of dimensionless numbers Lu, Bi_q, Bi_m, ε and Pn, are presented in Tables 7-12 and in Figures 11-16.

The dimensionless rates of change of temperature and mass of the bound matter (local and average) can be obtained by differentiation of equations (1.28), (1.29), (1.36) and (1.37) with respect to dimensionless time, thus:

$$\frac{dT}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} (C_{n2} \cos \nu_1 \mu_n x/R$$

$$- C_{n1} \cos \nu_2 \mu_n x/R)\, \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.40}$$

$$\frac{dT}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.41}$$

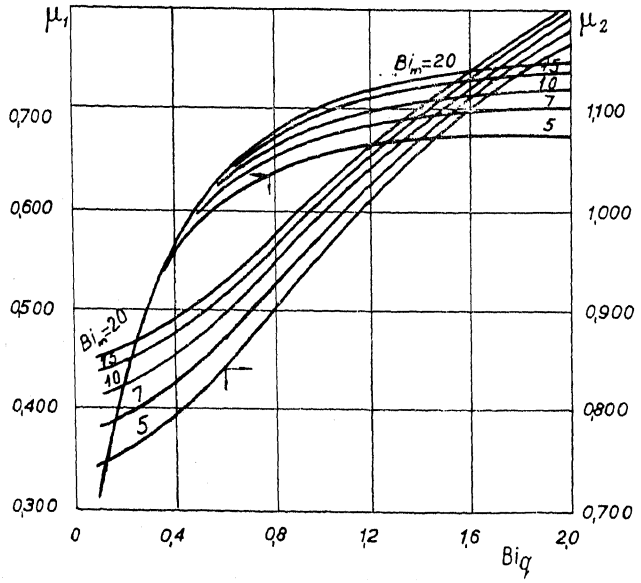

Fig. 4. Characteristic roots as a function of Bi$_m$ for an infinite plate (Bi$_q$ = 0,1 ÷ 2,0).

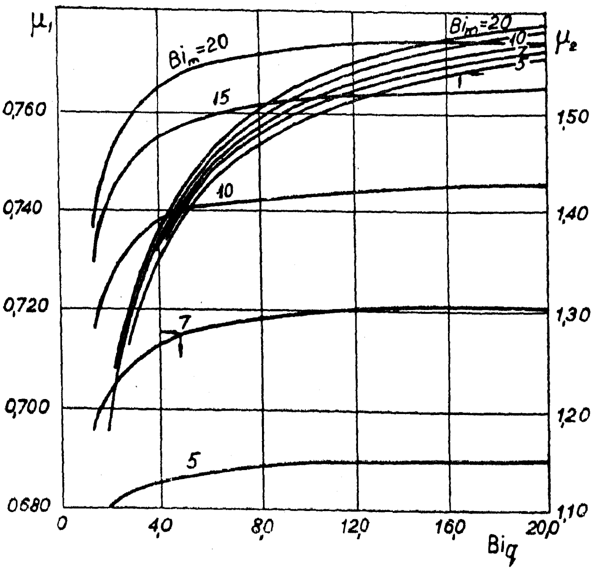

Fig. 5. Characteristic roots as a function of Bi$_m$ for an infinite plate (Bi$_q$ = 2 ÷ 20).

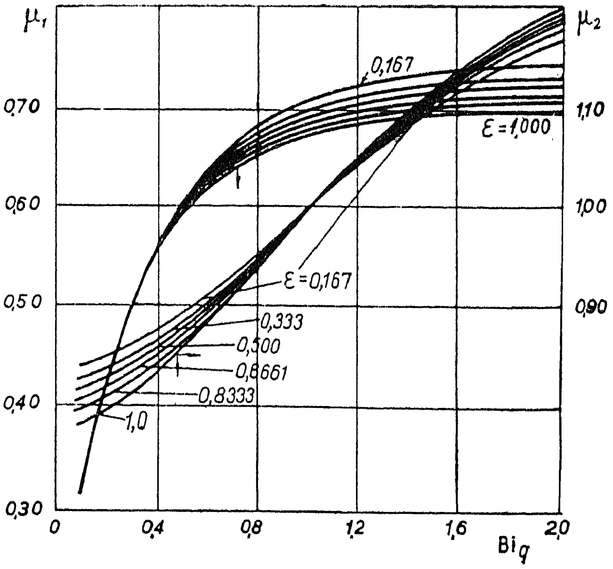

Fig. 6. Characteristic roots as a function of ε for an infinite plate ($\mathrm{Bi}_q = 0{,}1 \div 2{,}0$).

$$\frac{d\Theta}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} [C^*_{n1}(1 - \nu_2^2)\cos \nu_2 \mu_n \, x/R -$$

$$- C^*_{n2}(1 - \nu_1^2)\cos \nu_1 \mu_n x/R]\ \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.42}$$

$$\frac{d\overline{\Theta}}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D^*_n \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}). \tag{1.43}$$

Since $\mathrm{Fo} = \frac{a_q \tau}{R^2}$, the local and average heating and drying rates of a moist plate are inversely proportional to the square of its characteristic dimension (half thickness of the plate) and directly proportional to the coefficient of thermal diffusivity.

b) Sphere, Uniform Initial Conditions ($\Gamma = 2,\ \xi = r, f(r) = t_0 = \mathrm{const}$, $\varphi(r) = \theta_0 = \mathrm{const}$).

Let us introduce the following nomenclature:

$$L[rt(r, \tau)] = \int_0^{\infty} rt(r, \tau)\, e^{-s\tau}\, d\tau = \vartheta_L(r, s) = rt_L(r, s), \tag{1.44}$$

$$L\left[r\theta(r,\tau)\right]=\int\limits_0^\infty r\theta(r,\tau)\,e^{-s\tau}\,d\tau =$$

$$= u_L(r,s) = r\theta_L(r,s). \tag{1.45}$$

Applying the Laplace transformation equations (1.44) and (1.45) to equations (I) to (II) we get

$$\vartheta''_L(r,s) - \frac{s}{a_q}\,\vartheta_L(r,s) +$$

$$+\frac{\varepsilon\rho}{a_q c}\,s u_L(r,s) - \frac{\varepsilon\rho}{a_q c}\,r\theta_0 + \frac{rt_0}{a_q} = 0, \tag{1.46}$$

$$u''_L(r,s) - \frac{s}{a_m}\,u_L(r,s) + \delta\vartheta''_L(r,s) + \frac{r\theta_0}{a_m} = 0. \tag{1.47}$$

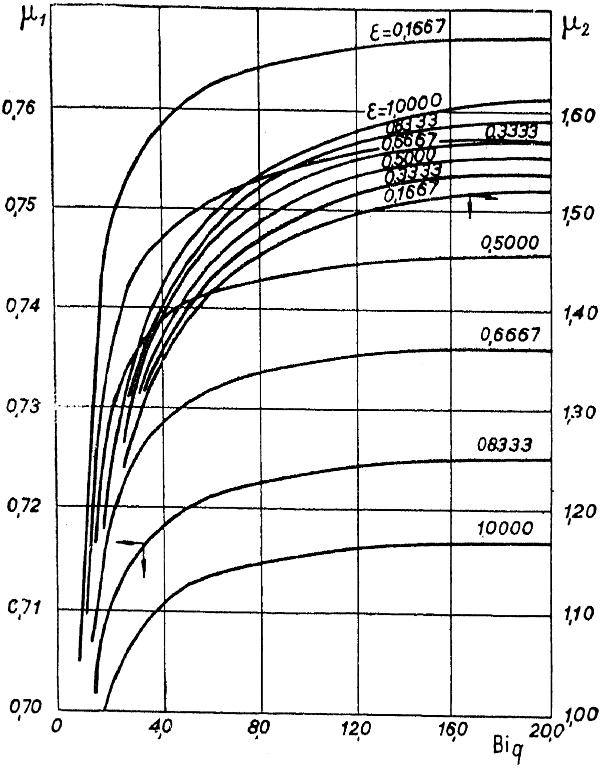

Fig. 7. The characteristic roots as a function of ε for an infinite plate (Bi$_q$ = 1—20).

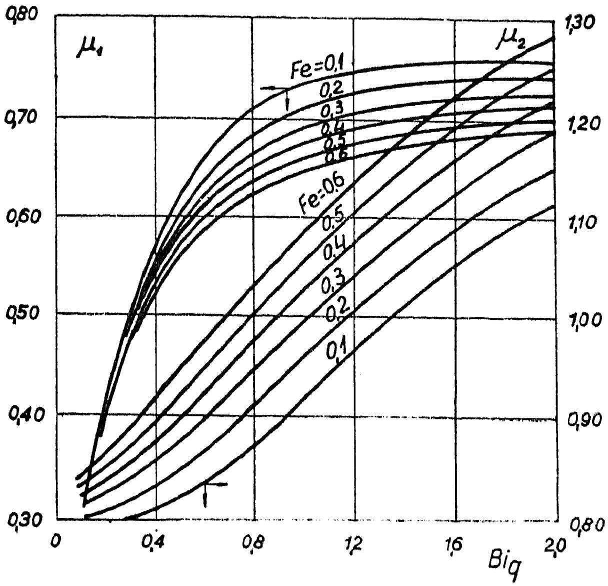

Fig. 8. Characteristic roots as a function of Fe for an infinite plate ($\mathrm{Bi}_q = 0{,}1$—$2{,}0$).

Equation (1.46) can be used to determine $u_L(r, s)$ and its second derivative $u_L''(r, s)$. The expressions found can then be substituted into equation (1.47). After transformation using the modified Laplace transform of temperature $\vartheta_L(r, s)$ we get an inhomogeneous ordinary differential equation of the fourth order:

$$\vartheta_L^{\mathrm{IV}}(r, s) - \left(\frac{s}{a_q} + \frac{\varepsilon\rho\delta\, s}{a_q c} + \frac{s}{a_m}\right)\vartheta_L''(r, s) + \frac{s^2}{a_q a_m}\,\vartheta_L(r, s) - \frac{s r t_0}{a_q a_m} = 0. \tag{1.48}$$

The general solution of equation (1.48) may be written as:

$$\vartheta_L(r, s) = \frac{r t_0}{s} + C_1\, e^{\sqrt{\frac{s}{a_q}}\,\nu_1 r} + C_2\, e^{\sqrt{\frac{s}{a_q}}\,\nu_2 r} + C_3\, e^{-\sqrt{\frac{s}{a_q}}\,\nu_1 r} + C_4\, e^{-\sqrt{\frac{s}{a_q}}\,\nu_2 r}, \tag{1.49}$$

where the $C_k\,(k = 1, 2, 3, 4)$ — are arbitrary constants and the ν_j are given by equation (1.5).

The solution for $u_L(r, s)$ can be found by substituting $\vartheta''_L(r, s)$ and $\vartheta_L(r\ s)$ into equation (1.46). Now we have

$$u_L(r, s) = \frac{r\theta_0}{s} + C_1\,\frac{c}{\varepsilon\rho}\,(1-\nu_1^2)\,e^{\sqrt{\frac{s}{a_q}}\,\nu_1 r} +$$

$$+ C_2\,\frac{c}{\varepsilon\rho}\,(1-\nu_2^2)\,e^{\sqrt{\frac{s}{a_q}}\,\nu_2 r} +$$

$$+ C_3\,\frac{c}{\varepsilon\rho}\,(1-\nu_1^2)\,e^{-\sqrt{\frac{s}{a_q}}\,\nu_1 r} + C_4\,\frac{c}{\varepsilon\rho}\,(1-\nu_2^2)\,e^{-\sqrt{\frac{s}{a_q}}\,\nu_2 r}. \qquad (1.50)$$

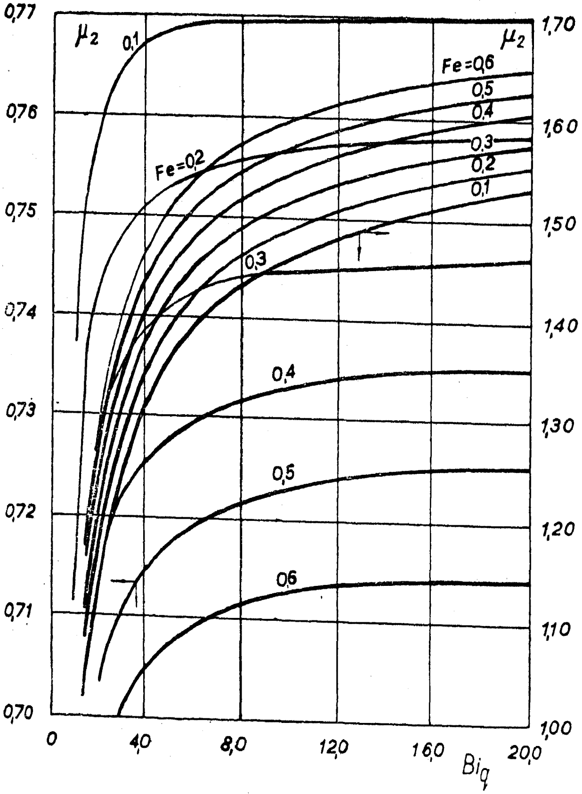

Fig. 9. Characteristic roots as a function of Fe for an infinite plate ($Bi_q = 1{,}0 \div 20$).

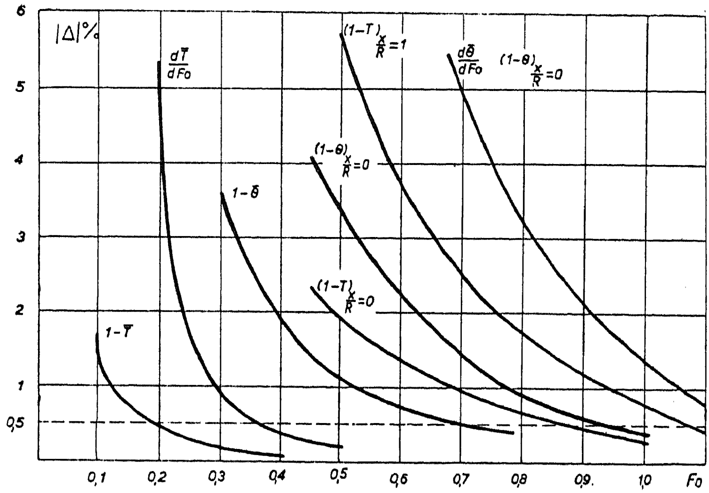

Fig. 10. Error introduced in the determination of $1-T; 1-\Theta; 1-\overline{T}; 1-\bar{\theta}; \frac{d\overline{T}}{d\mathrm{Fo}}; \frac{d\bar{\Theta}}{d\mathrm{Fo}}$ by neglecting the terms of the infinite series beginning with the third term.

The constants C_k are determined by the boundary conditions and the conditions of symmetry (III) to (V). The latter, after applying the Laplace transformation will be

$$t'_L(0, s) \text{ and } \theta'_L(0, s) = 0.$$

Because

$$r^2 \frac{dt_L}{dr} = \frac{d\vartheta_L}{dr} r - \vartheta_L \text{ and } t'_L(0, \tau) < \infty,$$

$$\lim_{r\to 0}\left[r^2 \frac{dt_L(r, s)}{dr}\right] =$$

$$= \lim_{r\to 0}\left[r \frac{d\vartheta_L(r, s)}{dr} - \vartheta_L(r, s)\right] = 0.$$

The conditions of symmetry for the modified temperature can therefore be rewritten:

$$\vartheta_L(0, s) = 0. \qquad (1.51)$$

We can write an analogue condition for the modified mass potential:

$$u_L(0, s) = 0. \tag{1.52}$$

It follows from equations (1.51) and (1.52) that $C_3 = -C_1$, $C_4 = -C_2$. This means that equations (1.49) and (1.51) may be somewhat simplified:

$$\theta_L(r, s) - \frac{rt_0}{s} = B_1 \operatorname{sh} \sqrt{\frac{s}{a_q}}\, \nu_1 r + B_2 \operatorname{sh} \sqrt{\frac{s}{a_q}}\, \nu_2 r, \tag{1.53}$$

$$u_L(r, s) - \frac{r\theta_0}{s} = B_1 \frac{c}{\varepsilon\rho} (1 - \nu_1^2) \operatorname{sh} \sqrt{\frac{s}{a_q}}\, \nu_1 r + B_2 \frac{c}{\varepsilon\rho} (1 - \nu_2^2) \operatorname{sh} \sqrt{\frac{s}{a_q}}\, \nu_2 r, \tag{1.54}$$

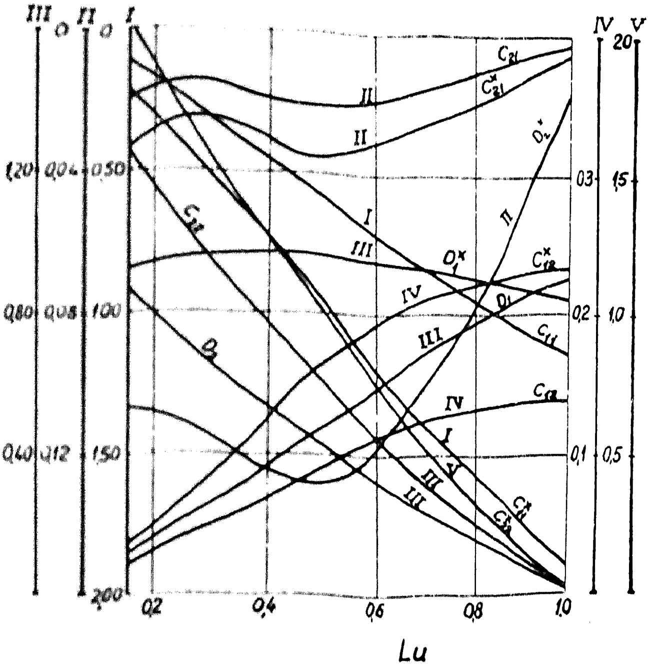

Fig. 11. Heat and mass transfer coefficients as a function of the Lu number.

or

$$t_L(r,s) - \frac{t_0}{s} = B_1 \frac{\operatorname{sh}\sqrt{\dfrac{s}{a_q}}\,\nu_1 r}{r} + B_2 \frac{\operatorname{sh}\sqrt{\dfrac{s}{a_q}}\,\nu_2 r}{r}, \tag{1.55}$$

$$\theta_L(r,s) - \frac{\theta_0}{s} = B_1 \frac{c}{\varepsilon\rho}(1-\nu_1^2)\frac{\operatorname{sh}\sqrt{\dfrac{s}{a_q}}\,\nu_1 r}{r} + B_2 \frac{c}{\varepsilon\rho}(1-\nu_2^2)\frac{\operatorname{sh}\sqrt{\dfrac{s}{a_q}}\,\nu_2 r}{r}. \tag{1.56}$$

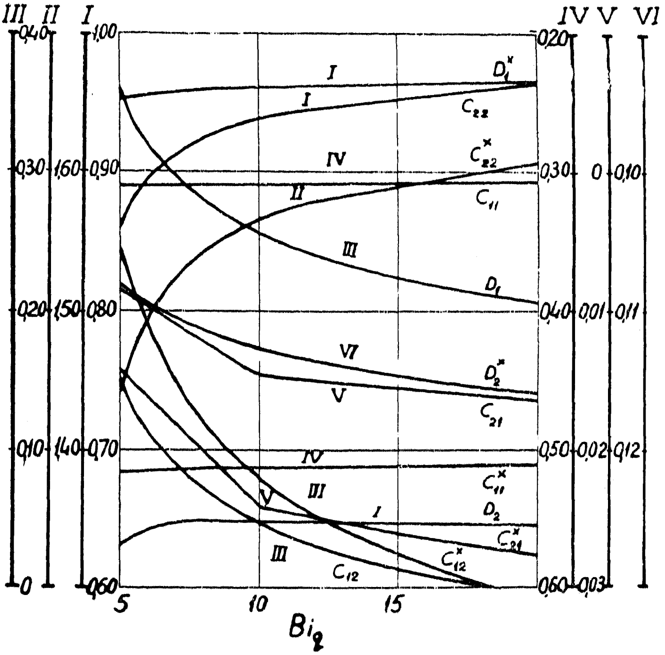

Fig. 12. Heat and mass transfer coefficients as a function of Bi_q.

Here $B_1 = 2C_1$ and $B_2 = 2C_2$ are new constants with respect to the coordinate and can be determined from boundary conditions (III) to (IV) which, after the application of the Laplace transformation, can be written as

$$-\lambda t'_L(R, s) + \alpha\left[\frac{t_c}{s} - t_L(R, s)\right] -$$

$$-(1-\varepsilon)\rho\beta\left[\theta_J(R, s) - \frac{\theta_p}{s}\right] = 0, \qquad (1.57)$$

$$\lambda_m \theta'_L(R, s) + \lambda_m \delta t'_L(R, s) +$$

$$+\beta\left[\theta_L(R, s) - \frac{\theta_p}{s}\right] = 0. \qquad (1.58)$$

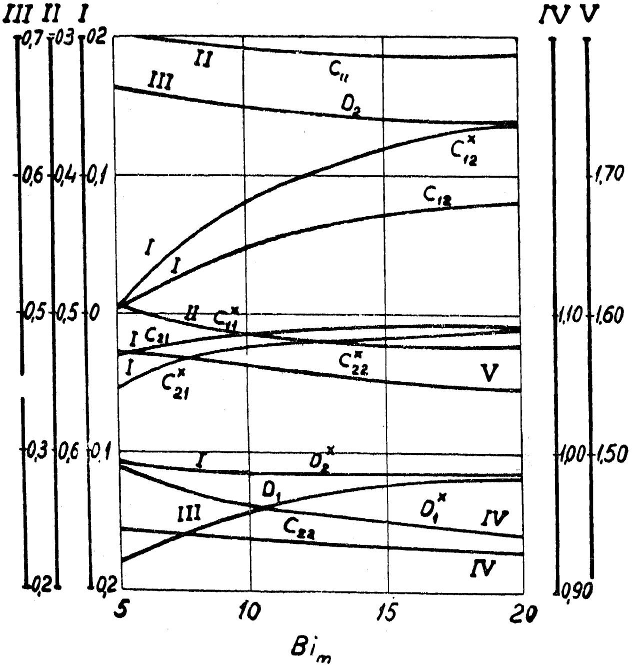

Fig. 13. Heat and mass transfer coefficients as a function of Bi_m.

Solutions (1.55) and (1.56) are completed by use of boundary conditions (1.57) and (1.58). Following simplification, the system of equations for B_1 and B_2 may be written in the following manner:

Table 7. Coefficients in the solution of the system of heat and mass transfer equations as a function of Lu.

(Ko = 1,2, Pn = 0,5, ε = 0,5, Bi_m = 10, Bi_q = 10)

Lu	C_{n1}		C_{n2}		C^*_{n1}		C^*_{n2}		D_n		D^*_n	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0,15	—0,1336	—0,01978	0,02312	1,258	—0,2226	—0,03298	0,03852	2,096	0,1144	0,8651	0,9240	—0,1070
0,3	—0,3080	—0,01452	0,04880	1,9378	—0,5134	—0,02420	0,08130	1,564	0,2570	0,6500	0,9161	—0,1137
0,5	—0,5840	—0,02162	0,09142	0,6064	—0,9732	—0,03606	0,1524	1,011	0,4821	0,4232	0,974	—0,1274
0,7	—0,8483	—0,01546	0,1270	0,3146	—1,415	—0,02574	0,2114	0,5246	0,6945	0,2213	0,9375	—0,0991
1,0	—1,135	-0,003402	0,1399	0,03438	—1,892	—0,00572	0,2334	0,0572	0,9009	0,02439	0,8435	—0,01538
2,0	—1,452	0,04048	0,1118	—0,1828	—2,420	0,06738	0,1864	—0,3044	1,100	—0,1227	0,6463	0,1328

Table 8. Infinite plate. Coefficients in the solution of the system of heat and mass transfer equations as a function of Bi_q.

Bi_q (Lu = 0,3, ε = 0,5, Ko = 1,2, Pn = 0,5, Bi_m = 10)

Bi_q	C_{n1}		C_{n2}		C^*_{n1}		C^*_{n2}		D_n		D^*_n	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
1,0	—0,2986	—0,009234	1,348	—0,3906	—0,4974	—0,01539	2,246	—0,6514	1,473	—0,3308	0,7652	0,08159
5,0	—0,3102	—0,008446	0,1563	0,8598	—0,5168	—0,01409	0,2606	1,434	0,3582	0,6291	0,9531	—0,1088
7,0	—0,3088	—0,01090	0,09294	0,9122	—0,5148	—0,01816	0,1550	1,521	0,2984	0,6491	0,9579	—0,1129
10	—0,3080	—0,01452	0,04880	0,9378	—0,5134	—0,02420	0,08130	1,564	0,2570	0,6500	0,9616	—0,1137
15	—0,3074	—0,01556	0,01517	0,9524	—0,5122	—0,02594	0,02526	1,588	0,2254	0,6472	0,9635	—0,1146
20	—0,3068	—0,01534	—0,002564	0,9630	—0,5116	—0,02724	-0,004274	1,606	0,2084	0,6473	0,9643	—0,1156

Table 9. Coefficients in the solution of the system of heat and mass transfer equations as a function of Bi_m.

($Lu = 0{,}3$, $\varepsilon = 0{,}5$, $Ko = 1{,}2$, $Pn = 0{,}5$, $Bi_q = 10$)

Bi_m	C_{n1}		C_{n2}		C^*_{n1}		C^*_{n2}		D_n		D^*_n	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0,1	—2,558	0,1000	16,65	0,3122	— 4,266	0,1667	27,74	0,5206	12,83	0,1939	—0,178	0,00417
0,5	—0,2500	0,02192	0,2408	0,5598	—0,4168	—0,03656	0,4018	0,9346	0,4660	0,5394	1,014	—0,02065
5	—0,2974	—0,03146	0,00369	0,9466	—0,4958	—0,05242	0,006154	1,578	0,2202	0,6635	0,9910	—0,1087
7	—0,3034	—0,02174	0,02590	0,9428	—0,5054	—0,03622	0,04320	1,572	0,2384	0,6570	0,9751	—0,1120
10	—0,3080	—0,01452	0,04880	0,9378	—0,5134	—0,02420	0,08130	1,564	0,2670	0,6500	0,9616	—0,1137
15	—0,3122	-0,009438	0,07318	0,9320	—0,5202	—0,01574	0,1219	1,554	0,2776	0,6423	0,9502	—0,1143
20	—0,3136	—0,006896	0,08182	0,9284	—0,5228	—0,01148	0,1364	1,548	0,2835	0,6378	0,9410	—0,1144

Table 10. Coefficients in the solution of the system of heat and mass transfer equations as a function of ε.

($Lu = 0{,}3$, $Ko = 1{,}2$, $Pn = 0{,}6$, $Bi_m = 10$, $Bi_q = 10$)

ε	C_{n1}		C_{n2}		C^*_{n1}		C^*_{n2}		D_n		D^*_n	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0,1667	—0,1173	-0,006474	0,08554	1,112	—0,5868	—0,03238	0,4276	5.562	0,1590	0,7636	0,990	—0,1433
0,3333	—0,2190	—0,01140	0,06714	1,018	—0,5472	—0,02850	0,1679	2,546	0,2128	0,7021	0,9753	—0,1268
0,5000	—0,3080	—0,01452	0,04880	0,9378	—0,5134	—0,02420	0,08130	1,564	0,2570	0,6500	0,9616	—0,1137
0,6667	—0,3872	—0,01751	0,03478	0,8638	—0,4840	—0,02190	0,04352	1,080	0,2979	0,6004	0,9484	—0,1019
0,8333	—0,4592	—0,02002	0,02734	0,7946	—0,4592	—0,02002	0,02734	0,7946	0,3401	0,5540	0,9399	—0,09164
1,0000	—0,5244	—0,01967	0,01669	0,7340	—0,4368	—0,01640	0,01391	0,6116	0,3738	0,5143	0,9301	—0,08296

Table 11. Coefficients in the solution of the system of heat and mass transfer equations as a function of Pn.

(Lu = 0,3, Ko = 1,2, ε = 0,5, Bi_m = 10, Bi_q = 10)

Pn	C_{n1}		C_{n2}		C_{n1}^*		C_{n2}^*		D_n		D_n^*	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0,1667	—0,3188	-0,005276	0,04618	0,9158	—0,5312	-0,008794	0,07694	1,527	0,2620	0,6353	0,9045	—0,03952
0,3333	—0,3132	-0.009060	0,04718	0,9288	—0,5220	-0.01509	0,07860	1,549	0,2590	0,6452	0,9336	—0,07796
0,5000	—0,3080	-0,01452	0,04880	0,9378	—0,5134	-0,02420	0,08130	1,564	0,2570	0,6500	0,9616	—0,1137
0,6667	—0,3028	-0,01809	0,04882	0,9524	—0,5048	-0,03016	0,08138	1,588	0,2532	0,6600	0,9865	—0,1500
0,8333	—0,2978	-0,02246	0,04830	0,9694	—0,4964	-0,03742	0,08048	1,616	0,2493	0,6709	1,010	—0,1862

Table 12. Coefficients in the solution of the system of heat and mass transfer equations as a function of Ko.

(Lu = 0,3, Pn = 0,5, ε = 0,5. Bi_m = 10, Bi_q = 10)

Ko	C_{n1}		C_{n2}		C_{n1}^*		C_{n2}^*		D_n		D_n^*	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0,4	—0,1172	-0,006408	0,01408	1,133	—0,5862	—0,03204	0,07050	5,670	0,09371	0,7859	0,9986	—0,1470
0,8	—0,2186	-0,01032	0,03190	1,029	—0,5466	—0,02580	0,07982	2,574	0,1799	0,7148	0,9777	—0,1293
1,2	—0,3080	-0,01452	0,04880	0,9378	—0,5134	—0,02420	0,08130	1,564	0,2570	0,6500	0,9616	—0,1137
1,6	—0,3876	-0,01630	0,06270	0,8658	—0,4844	—0,02036	0,07842	1,082	0,3244	0,6001	0,9466	—0,1022
2,0	—0,4592	-0,01827	0,07526	0,8072	—0,4592	—0,01827	0,07526	0,8072	0,3850	0,5586	0,9338	—0,09302

$$B_1\widetilde{Q}_1 + B_2\widetilde{Q}_2 - \frac{t_c - t_0}{s} + (1-\varepsilon)\frac{\rho\beta}{\alpha}\,\frac{\theta_0 - \theta_p}{s} = 0,$$

$$B_1\widetilde{P}_1 + B_2\widetilde{P}_2 + \frac{\mathrm{Fe}}{\delta}\,\frac{\theta_0 - \theta_p}{s} = 0. \qquad (1.59)$$

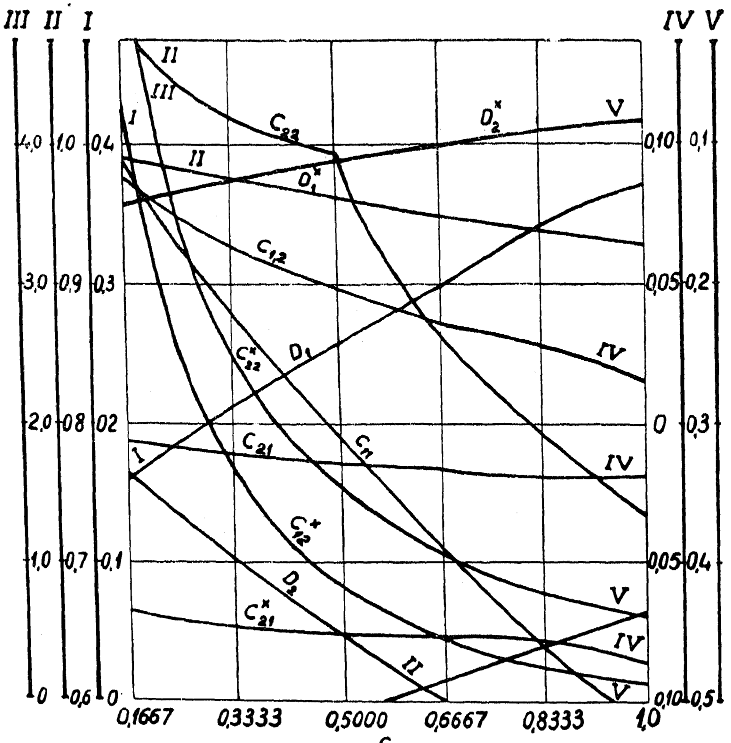

Fig. 14. Heat and mass transfer coefficients as a function of ε.

We can now determine the coefficients B_j. We have

$$B_1 = -\frac{(t_c - t_0)\widetilde{P}_2 + \frac{\varepsilon\rho}{c}\,(\theta_0 - \theta_p)\,(\widetilde{Q}_2 - K_1\widetilde{P}_2)}{s(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2)},$$

$$B_2 = \frac{(t_c - t_0)\widetilde{P}_1 + \frac{\varepsilon\rho}{c}\,(\theta_0 - \theta_p)\,(\widetilde{Q}_1 - K_1\widetilde{P}_1)}{s(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2)},$$

where

$$\widetilde{Q}_j = \frac{1}{\mathrm{Bi}_q}\sqrt{\frac{s}{a_q}}\,\nu_j R\,\frac{\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_j R}{R} +$$

$$+\left[1-\frac{1}{\mathrm{Bi}_q}-(1-\nu_j^2)K_1\right]\frac{\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_j R}{R}, \qquad (1.60)$$

$$\widetilde{P}_j = \left[(1-\nu_j^2)+\mathrm{Fe}\right]\frac{1}{\mathrm{Bi}_m}\sqrt{\frac{s}{a_q}}\,\nu_j R\,\frac{\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_j R}{R} +$$

$$+\left[(1-\nu_j^2)-\frac{(1-\nu_j^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\right]\frac{\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_j R}{R}, \qquad (1.61)$$

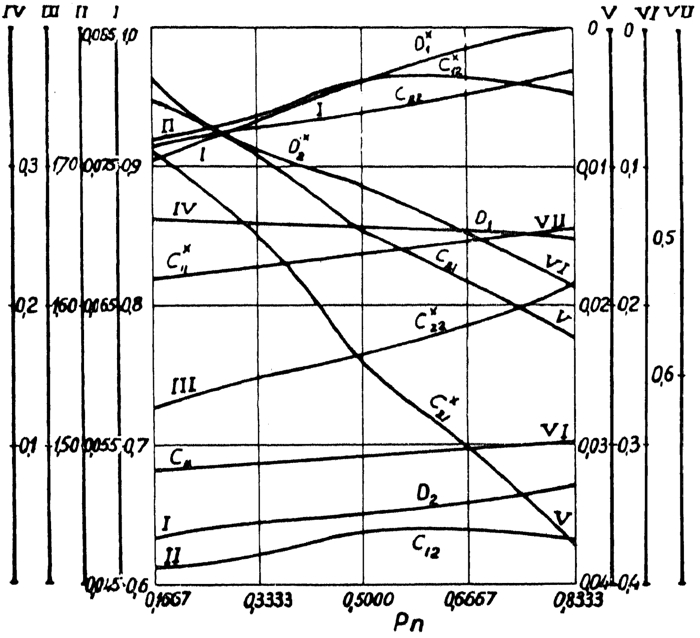

Fig. 15. Heat and mass transfer as a function of Pn.

Thus, the Laplace transform solutions for the modified transfer potentials (1.53) and (1.54) are:

$$\vartheta_L(r,s) - \frac{rt_0}{s} = (t_c - t_0)\left\{\frac{\widetilde{P}_1 + \varepsilon \mathrm{Ko}\,(\widetilde{Q}_1 - K_1\widetilde{P}_1)}{s\,(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2)}\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r - \frac{\widetilde{P}_2 + \varepsilon \mathrm{Ko}(\widetilde{Q}_2 - K_1\widetilde{P}_2)}{s\,(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2)}\,\mathrm{sh}\sqrt{\frac{s}{a_q}\nu_1 r}\right\} = (t_c - t_0)\frac{\Phi_1(s)}{\psi_1(s)}\,, \qquad (1.62)$$

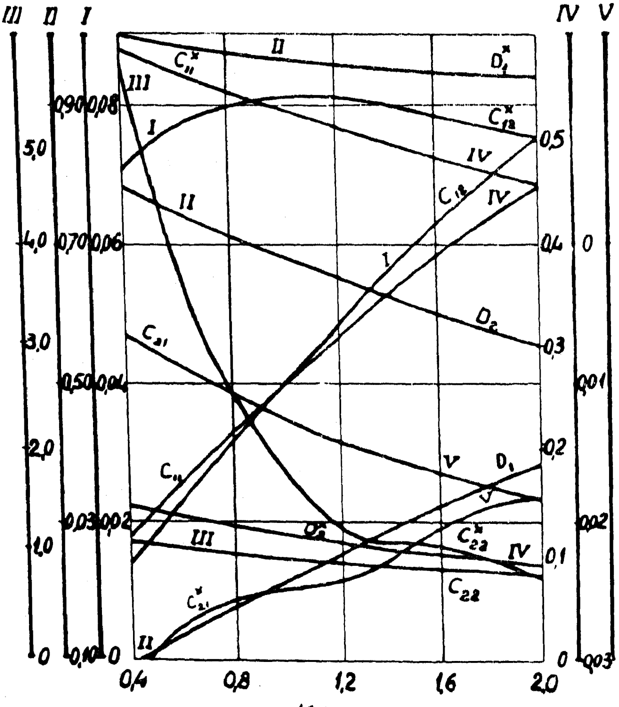

Fig. 16. Heat and mass transfer coefficients as a function of the Ko number.

$$\frac{r\,\theta_0}{s} - u_L(r,s) = (\theta_0 - \theta_p)\left\{\frac{\widetilde{Q}_2 + \dfrac{1}{\varepsilon\,\mathrm{Ko}}\widetilde{P}_2 - K_1\widetilde{P}_2}{s(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2)}\right. \times$$

$$\times (1-\nu_1^2)\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r - \frac{\widetilde{Q}_1 + \frac{1}{\varepsilon \mathrm{Ko}}\widetilde{P}_1 - K_1\widetilde{P}_1}{s(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2)} \times$$

$$\left.\times (1-\nu_2^2)\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r\right\} = (\theta_0 - \theta_p)\frac{\Phi_2(s)}{\psi_1(s)}\,. \qquad (1.63)$$

The numerator and denominator in equations (1.62) and (1.63) are not generalized polynomials with respect to s; however, they can be reduced to such form by dividing both parts, i.e. $\Phi_1(s)$ and $\psi_1(s)$, $\Phi_2(s)$ and $\psi_1(s)$ by $\sqrt{\frac{s}{a_a}}\ \sqrt{\frac{s}{a_q}}$. Analysis of the cited equations also shows that inversion of the transform, with the exception of the zeroth root $(s = 0)$, can be effected by using the usual theorem of division, that is, without the transformation of either $\Phi_1(s)$ or $\Phi_2(s)$ into a generalized polynomial $\Phi_{01}(s)$ or $\Phi_{02}(s)$ and $\psi_1(s)$ into $\psi_0(s)$. The roots $\psi_1(s)$ can now be found:

$$\psi_1(s) = s(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2) = 0.$$

We have: a) $s = s_0 = 0$ (zeroth root),
b) $s = s_n$, where s_n satisfies the equation:

$$\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2 = 0. \qquad (1.64)$$

In equation (1.64) we replace the hyperbolic functions by trigonometric functions and denote $i\sqrt{\frac{s}{a_q}}\,R$ by μ; then for s_n we get the following expression:

$$s_n = -\frac{a_q\,\mu_n^2}{R^2}\,,$$

where μ_n, can be determined from

$$Q_{n2}P_{n1} - Q_{n1}P_{n2} = 0. \qquad (1.65)$$

Here,

$$Q_{nj} = \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n +$$

$$+ \left[1 - \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2) K_1 \right] \sin \nu_j \mu_n, \tag{1.66}$$

$$P_{nj} = \left[(1 - \nu_j^2) + \mathrm{Fe} \right] \frac{\nu_j \mu_n}{\mathrm{Bi}_m} \cos \nu_j \mu_n + \left[(1 - \nu_j^2) - \right.$$

$$\left. - \frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m} \right] \sin \nu_j \mu_n. \tag{1.67}$$

Thus, the inverse of the transforms may be expressed as follows:

$$L^{-1} \left[\frac{\Phi_1(s)}{\psi_1(s)} \right] = \frac{\Phi_{01}(0)}{\psi'_0(0)} + \sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi'_1(s_n)} \exp(s_n \tau), \tag{1.68}$$

$$L^{-1} \left[\frac{\Phi_2(s)}{\psi_1(s)} \right] = \frac{\Phi_{02}(0)}{\psi'_0(0)} + \sum_{n=1}^{\infty} \frac{\Phi_2(s_n)}{\psi'_1(s_n)} \exp(s_n \tau). \tag{1.69}$$

For the determination of the zeroth term, we convert equation (1.68) to represent a ratio of two generalized polynomials. The latter are

$$\Phi_{01}(s) = \left\{ \left[\frac{(1 - \nu_1^2) + \mathrm{Fe}}{\mathrm{Bi}_m} \nu_1 \left(1 + \frac{1}{2!} \frac{s}{a_q} \nu_1^2 R^2 + \dots \right) + \right. \right.$$

$$\left. + \left[(1 - \nu_1^2) - \frac{(1 - \nu_1^2) + \mathrm{Fe}}{\mathrm{Bi}_m} \right] \left(\nu_1 + \frac{1}{3!} \frac{s}{a_q} \nu_1^3 R^2 + \dots \right) \right] +$$

$$+ \varepsilon \mathrm{Ko} \left[\frac{\nu_1}{\mathrm{Bi}_q} \left(1 + \frac{1}{2!} \frac{s}{a_q} \nu_1^2 R^2 + \dots \right) + \right.$$

$$+ \left[1 - \frac{1}{\mathrm{Bi}_q} + (1 - \nu_1^2) K_1 \right] \times$$

$$\times \left(\nu_1 + \frac{1}{3!} \nu_1^3 R^2 \frac{s}{a_q} + \dots \right) -$$

$$-K_1\Bigg(\frac{(1-\nu_1^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\,\nu_1\left(1+\frac{1}{2!}\,\frac{s}{a_q}\,\nu_1^2\,R^2+\dots\right)+$$

$$+\left[(1-\nu_1^2)-\frac{(1-\nu_1^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\right]\times$$

$$\times\left(\nu_1+\frac{1}{3!}\,\frac{s}{a_q}\,\nu_1^3\,R^2+\dots\right)\Bigg)\Bigg]\Bigg\}\times$$

$$\times\left(\nu_2\,r+\frac{1}{3!}\,\frac{s}{a_q}\,\nu_2^3\,r^3+\dots\right)-$$

$$-\Bigg\{\Bigg[\frac{(1-\nu_2^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\,\nu_2\left(1+\frac{1}{2!}\,\frac{s}{a_q}\,\nu_2^2\,R^2+\dots\right)+$$

$$+\left[(1-\nu_2^2)-\frac{(1-\nu_2^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\right]\times$$

$$\times\left(\nu_2+\frac{1}{3!}\,\frac{s}{a_q}\,\nu_2^3\,R^2+\dots\right)\Bigg]+$$

$$+\varepsilon\,\mathrm{Ko}\Bigg[\frac{\nu_2}{\mathrm{Bi}_q}\left(1+\frac{1}{2!}\,\frac{s}{a_q}\,\nu_2^2R^2+\dots\right)+$$

$$+\left[1-\frac{1}{\mathrm{Bi}_q}+(1-\nu_2^2)\,K_1\right]\times$$

$$\times\left(\nu_2+\frac{1}{3!}\,\nu_2^3\,R^2\,\frac{s}{a_q}+\dots\right)-$$

$$-K_1\Bigg(\frac{(1-\nu_2^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\,\nu_2\left(1+\frac{1}{2!}\,\frac{s}{a_q}\,\nu_2^2\,R^2+\dots\right)+$$

$$+\left[(1-\nu_2^2)-\frac{(1-\nu_2^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\right]\times$$

$$\times\left(\nu_2+\frac{1}{3!}\,\frac{s}{a_q}\,\nu_2^3\,R^2+\dots\right)\Bigg)\Bigg]\Bigg\}\times$$

$$\times\left(\nu_1 r+\frac{1}{3!}\,\frac{s}{a_q}\,\nu_1^3\,r^3+\dots\right),$$

$$\psi_0(s)=s\Bigg[\Bigg\{\frac{\nu_2}{\mathrm{Bi}_q}\Bigg(1+\frac{1}{2!}\frac{s}{a_q}\nu_2^2R^2+\dots\Bigg)+$$

$$+\Bigg[1-\frac{1}{\mathrm{Bi}_q}+K_1(1-\nu_2^2)\Bigg]\times$$

$$\times\Bigg(\nu_2+\frac{1}{3!}\nu_2^3R^2\frac{s}{a_q}+\dots\Bigg)\Bigg\}\times$$

$$\times\Bigg\{\frac{(1-\nu_1^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\nu_1\Bigg(1+\frac{1}{2!}\frac{s}{a_q}\nu_1^2R^2+\dots\Bigg)+$$

$$+\Bigg[(1-\nu_1^2)-\frac{(1-\nu_1^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\Bigg]\times$$

$$\times\Bigg(\nu_1+\frac{1}{3!}\frac{s}{a_q}\nu_1^3R^2+\dots\Bigg)\Bigg\}-$$

$$-\Bigg\{\frac{\nu_1}{\mathrm{Bi}_q}\Bigg(1+\frac{1}{2!}\frac{s}{a_q}\nu_1^2R^2+\dots\Bigg)+$$

$$+\Bigg[1-\frac{1}{\mathrm{Bi}_q}+K_1(1-\nu_1^2)\Bigg]\times$$

$$\times\Bigg(\nu_1+\frac{1}{3!}\frac{s}{a_q}\nu_1^3R^2+\dots\Bigg\}\times$$

$$\times\Bigg\{\frac{(1-\nu_2^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\nu_2\Bigg(1+\frac{1}{2!}\frac{s}{a_q}\nu_2^2R^2+\dots\Bigg)+$$

$$+\Bigg[(1-\nu_2^2)-\frac{(1-\nu_2^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\Bigg]\times$$

$$\times\Bigg(\nu_2+\frac{1}{3!}\frac{s}{a_q}\nu_2^3R^2+\dots\Bigg)\Bigg\}\Bigg].$$

Now, the zero term of equation (1.68) may be expressed as

$$\lim_{s\to 0}\frac{\Phi_{01}(s)}{\psi_0'(s)}=r.$$

For the remaining roots we find

$$\psi_1'(s_n) = \frac{\mu_n}{2R^2}\psi_n,$$

where

$$\psi_n = \nu_1 P_{n2}A_{n1} + \nu_2 Q_{n1}B_{n2} - \nu_2 P_{n1}A_{n2} - \nu_1 Q_{n2}B_{n1}, \tag{1.70}$$

$$A_{nj} = \left[1 + (1-\nu_j^2)K_1\right]\cos\nu_j\mu_n - \\ - \frac{\nu_j\mu_n}{\mathrm{Bi}_q}\sin\nu_j\mu_n, \tag{1.71}$$

$$B_{ni} = (1-\nu_j^2)\cos\nu_j\mu_n - \\ - \left[(1-\nu_j^2) + \mathrm{Fe}\right]\frac{\nu_j\mu_n}{\mathrm{Bi}_m}\sin\nu_j\mu_n. \tag{1.72}$$

The term for the non-zero roots of equation (1.68) is

$$\sum_{n=1}^{\infty}\frac{\Phi_1(s_n)}{\psi_1'(s_n)}e^{s_n\tau} = -\sum_{n=1}^{\infty}\left(C_{n1}R\sin\nu_2\mu_n\frac{r}{R} - \right. \\ \left. - C_{n2}R\sin\nu_1\mu_n\frac{r}{R}\right)\exp\left(-\mu_n^2\,\mathrm{Fo}\right),$$

$$C_{nj} = 2\,\frac{P_{nj} + \varepsilon\,\mathrm{Ko}\,(Q_{nj} - K_1P_{nj})}{\mu_n\psi_n}. \tag{1.73}$$

Thus, the solution of the unsteady heat transfer potential field assumes the following form:

$$\frac{rt(r,\tau) - rt_0}{t_c - t_0} = r - \sum_{n=1}^{\infty}\left(C_{n1}R\sin\nu_2\mu_n\frac{r}{R} - \right. \\ \left. - C_{n2}R\sin\nu_1\mu_n\frac{r}{R}\right)\exp\left(-\mu_n^2\,\mathrm{Fo}\right). \tag{1.74}$$

The method for inverting equation (1.69) is analogous to the foregoing:

$$\frac{r\theta_0 - r\theta(r,\tau)}{\theta_0 - \theta_p} = r -$$

$$-\sum_{n=1}^{\infty}\left[C_{n2}^{*}(1-\nu_1^2)R\sin\nu_1\mu_n\frac{r}{R}-\right.$$

$$\left.-C_{n1}^{*}(1-\nu_2^2)R\sin\nu_2\mu_n\frac{r}{R}\right]\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (1.75)$$

$$C_{nj}^{*}=\frac{1}{\varepsilon\,\mathrm{Ko}}C_{nj}. \qquad (1.76)$$

The final solution of differential equations for internal heat and mass transfer in a sphere, with boundary conditions of the third kind and uniform initial conditions, is obtained by combining equations (1.65) to (1.67) and (1.70) to (1.76):

$$T=\frac{t(r,\tau)-t_0}{t_c-t_0}=1-\sum_{n=1}^{\infty}\left(C_{n1}\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}}-\right.$$

$$\left.-C_{n2}\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}}\right)\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (1.77)$$

$$\Theta=\frac{\theta_0-\theta(r,\tau)}{\theta_0-\theta_p}=1-\sum_{n=1}^{\infty}\left[C_{n2}^{*}(1-\nu_1^2)\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}}-\right.$$

$$\left.-C_{n1}^{*}(1-\nu_2^2)\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}}\right]\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (1.78)$$

where

$$C_{nj}=2\,\frac{P_{nj}+\varepsilon\,\mathrm{Ko}(Q_{nj}-K_1P_{nj})}{\mu_n\psi_n},$$

$$C_{nj}^{*}=\frac{1}{\varepsilon\,\mathrm{Ko}}C_{nj},$$

$$\nu_j^2 = \frac{1}{2}\left[\left(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\right)+\right.$$

$$\left.+(-1)^j\sqrt{\left(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\right)^2-\frac{4}{\mathrm{Lu}}}\,\right],$$

$$\psi_n=\nu_1 P_{n2} A_{n1}+\nu_2 Q_{n1} B_{n2}-\nu_1 P_{n1} A_{n2}-\nu_1 Q_{n2} B_{n1},$$

$$A_{nj}=\left[1+(1-\nu_j^2)K_1\right]\cos\nu_j\mu_n-\frac{\nu_j\mu_n}{\mathrm{Bi}_q}\sin\nu_j\mu_n,$$

$$B_{nj}=(1-\nu_j^2)\cos\nu_j\mu_n-\left[(1-\nu_j^2)+\mathrm{Fe}\right]\frac{\nu_j\mu_n}{\mathrm{Bi}_m}\sin\nu_j\mu_n,$$

$$Q_{nj}=\frac{\nu_j\mu_n}{Bi_q}\cos\nu_j\mu_n+$$

$$+\left[1-\frac{1}{\mathrm{Bi}_q}+(1-\nu_j^2)K_1\right]\sin\nu_j\mu_n,$$

$$P_{nj}=\left[(1-\nu_j^2)+\mathrm{Fe}\right]\frac{\nu_j\mu_n}{\mathrm{Bi}_m}\cos\nu_j\mu_n+$$

$$+\left[(1-\nu_j^2)-\frac{(1-\nu_j^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\right]\sin\nu_j\mu_n.$$

μ_n are the roots of the characteristic equation (1.65)

$$Q_{n2}P_{n1}-Q_{n1}P_{n2}=0.$$

Roots of the characteristic equation, which are necessary for heat and mass transfer calculations, can be obtained by a graphical-analytical method. As in the case of a plate, we transform equation (1.65) by denoting

$$M=P_1\frac{\sin\nu_2\mu}{\mu^{-1}}-P_2\frac{\sin\nu_1\mu}{\mu^{-1}},$$

$$N=P_2\left[\nu_1\mu\cos\nu_1\mu+\frac{1-\varepsilon}{\varepsilon}\mathrm{LuBi}_m(1-\nu_1^2)\sin\nu_1\mu\right]-$$

$$-P_1\left[\nu_2\mu\cos\nu_2\mu+\frac{1-\varepsilon}{\varepsilon}\mathrm{LuBi}_m(1-\nu_2^2)\sin\nu_2\mu\right],$$

and we obtain

$$\Phi_{\mu} = \frac{M}{N} = \frac{\mu}{\mathrm{Bi}_q - 1}\,. \tag{1.79}$$

The roots μ_n are determined from the intersection of the curves $\Phi = \Phi_{\mu}(\mu)$ and the straight line $y = \frac{\mu}{\mathrm{Bi}_q - 1}$. The shape of the curves and the method of determining the roots are given in Figure 17. The values of several first characteristic roots are given in Table 13 and also in Figures 18 and 19.

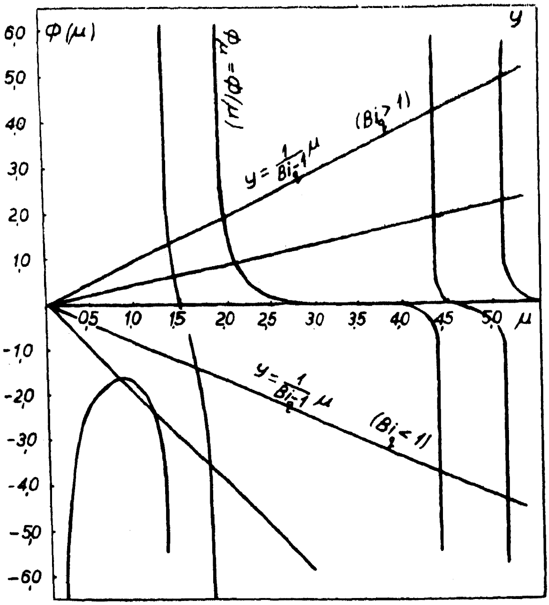

Fig. 17. Graphical method for the determination of roots of the characteristic equation $\Phi(\mu) = \frac{1}{\mathrm{Bi}_q - 1}\mu$ for a sphere.

The values of μ_n rapidly increase with increasing values of the index n. As a result, the infinite series in equations (1.77) and (1.78) converge at a fairly rapid rate, so that a certain value of Fo just two terms of the series will be sufficient. The starting values of Fo for simplified calculations of heat and mass transfer in a sphere are given in Table 6. Comparison of these data with corresponding data for an infinite plate shows that the starting Fo values for a

Table 13. Roots of the characteristic equation $\Phi(\mu)=\frac{1}{\mathrm{Bi}_q-1}\mu$ for a sphere ($\mathrm{Lu}=0{,}3$, $\mathrm{KoPn}=0{,}6$, $\varepsilon=0{,}5$, $\mathrm{Bi}_m=10$)

Bi_q	μ_1	μ_2	μ_3	μ_4
for $\mathrm{Bi}_q<1{,}0$				
0,10			1,756	5,076
0,15			1,760	5,079
0,20			1,763	5,081
0,25			1,766	5,082
0,30			1,769	
0,35			1,772	
0,40	0,970	1,125	1,776	
0,45	0,879	1,183	1,781	
0,50	0,811	1,217	1,787	
0,55	0,749	1,240	1,793	
0,60	0,695	1,257	1,800	
0,70	0,584	1,285	1,865	
0,80	0,462	1,315		
0,90	0,343			
for $\mathrm{Bi}_q>1{,}0$				
1,5	1,389	1,917	4,401	5,088
2,0	1,408	1,969	4,402	5,093
3,0	1,431	2,058	4,404	5,100
4,0	1,447	2,126	4,406	5,110
5,0	1,455	2,191	4,408	5,115
6,0	1,460	2,247	4,410	5,121
7,0	1,464	2,295	4,411	5,127
8,0	1,467	2,337	4,412	5,133
9,0	1,469	2,370	4,413	5,141
10	1,471	2,405	4,415	5,149
15	1,475	2,526	4,423	5,186
20	1,478	2,641	4,430	5,218
30	1,481	2,788	4,441	5,273
40	1,482	2,855	4,450	5,320
50	1,483	2,900	4,459	5,387
∞	1,486	2,960	4,518	5,881

sphere are considerably lower than for a plate. The heat and mass transfer coefficients for a sphere are given in Table 14. The same data are presented graphically in Figure 20.

The average values of heat and mass transfer potentials in a sphere are obtained from

$$\bar{t}(\tau)=\frac{3}{R^3}\int_0^R r^2 t(r,\tau)dr,\qquad \bar{\Theta}(\tau)=\frac{3}{R^3}\int_0^R r^2\,\Theta(r,\tau)\,dr.$$

The average dimensionless transfer potentials are

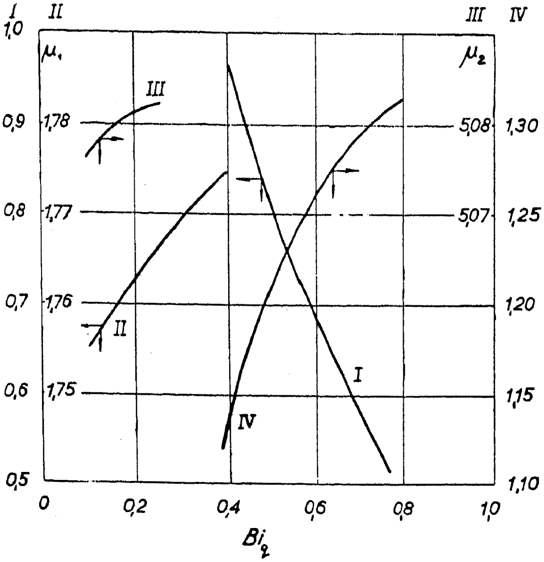

Fig. 18. Characteristic roots as a function of Bi_q for a sphere ($\mathrm{Bi}_q < 1,0$).

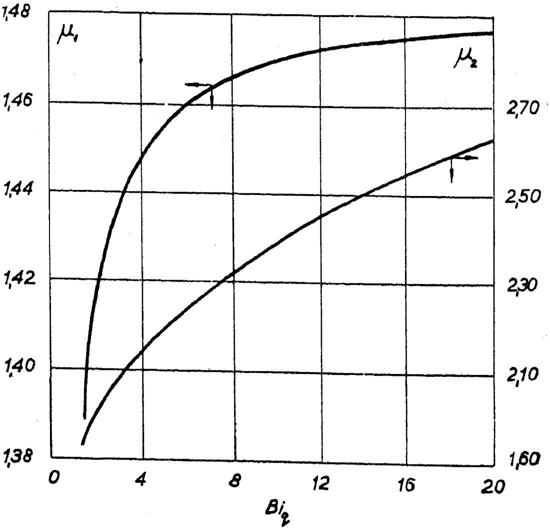

Fig. 19. Characteristic roots as a function of Bi_q for a sphere ($\mathrm{Bi}_q > 1,0$).

Table 14. Coefficients of the system of heat and mass transfer equations as a function of Bi_q.

(Lu=0,3, ε=0,5, Ko=1,2, Pn=0,5, Bi_m=10)

Bi_q	C_{n1}		C_{n2}		C^*_{n1}	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
5,0	0,1679	—0,1720	—0,1643	—0,9540	0,2799	—0,2867
7,0	0,1664	—0,1229	—0,09866	—0,7322	0,2774	—0,2049
10	0,1651	—0,08956	—0,05284	—0,5456	0,2752	—0,1493
15	0,1642	—0,06734	—0,01965	—0,3970	0,2737	—0,1123
20	0,1636	—0,05408	—0,004038	—0,3008	0,2727	—0,09015

Bi_q	C^*_{n2}		D_n		D^*_n	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
5,0	—0,2739	—1,590	0,3757	1,218	0,8343	—0,3637
7,0	—0,1645	—1,221	0,2987	0,9520	0,8334	—0,1940
10	—0,08808	—0,9095	0,2444	0,7161	0,8314	—0,0878
15	—0,03276	—0,6618	0,2052	0,5226	0,8315	—0,02132
20	—0,006731	—0,5014	0,1860	0,3943	0,8288	0,01378

$$\overline{T}=\frac{\overline{t}(\tau)-t_0}{t_c-t_0}=1-\sum_{n=1}^{\infty}D_n\exp(-\mu_n^2\,\mathrm{Fo}),\tag{1.80}$$

$$\overline{\Theta}=\frac{\theta_0-\overline{\theta}(\tau)}{\theta_0-\theta_p}=1-\sum_{n=1}^{\infty}D^*_n\exp(-\mu_n^2\,\mathrm{Fo}).\tag{1.81}$$

Here

$$D_n=D_{n1}-D_{n2},\quad D^*_n=D^*_{n2}-D^*_{n1},\tag{1.82}$$

$$D_{n1}=\frac{3(\sin\nu_2\mu_n-\nu_2\mu_n\cos\nu_2\mu_n)}{\nu_2^2\,\mu_n^2}C_{n1},$$

$$D_{n2}=\frac{3(\sin\nu_1\mu_n-\nu_1\mu_n\cos\nu_1\mu_n)}{\nu_1^2\,\mu_n^2}C_{n2},$$

$$D^*_{n1}=\frac{3(\sin\nu_2\mu_n-\nu_2\mu_n\cos\nu_2\mu_n)}{\nu_2^2\,\mu_n^2}(1-\nu_2^2)C^*_{n1},$$

$$D^*_{n2}=\frac{3(\sin\nu_1\mu_n-\nu_1\mu_n\cos\nu_1\mu_n)}{\nu_1^2\,\mu_n^2}(1-\nu_1^2)C^*_{n2}.$$

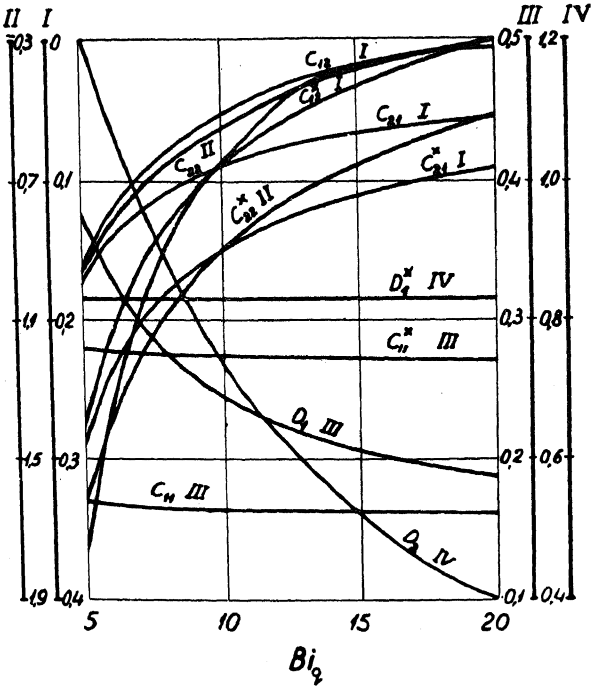

Fig. 20. Heat and mass transfer coefficients as a function of Bi_q for a sphere.

The coefficients D_n and D_n^* as functions of Bi_q are given in Table 14 and Figure 20.

The dimensionless rates of change of the potentials are obtained by differentiating equations (1.77) and (1.78) with respect to the dimensionless time. The average values can be obtained by similar differentiation of equations (1.80) and (1.81):

We have

$$\frac{dT}{d\mathrm{Fo}} = \sum_{n=1}^{\infty}\left(C_{n1}\frac{\sin \nu_2 \mu_n r/R}{r/R} - \right.$$

$$\left. - C_{n2}\frac{\sin \nu_1 \mu_n r/R}{r/R}\right)\mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}), \qquad (1.83)$$

$$\frac{d\bar{T}}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}), \qquad (1.84)$$

$$\frac{d\Theta}{d\text{Fo}} = \sum_{n=1}^{\infty}\left[C^{*}_{n2}(1-\nu_1^2)\,\frac{\sin \nu_1 \mu_n r/R}{r/R} - \right.$$

$$\left. - C^{*}_{n1}\,(1-\nu_2^2)\,\frac{\sin \nu_2 \mu_n r/R}{r/R}\right] \mu_n^2 \exp(-\mu_n^2\,\text{Fo}), \tag{1.85}$$

$$\frac{d\bar{\Theta}}{d\text{Fo}} = \sum_{n=1}^{\infty} D^{*}_{n}\;\mu_n^2 \exp(-\mu_n^2\,\text{Fo}). \tag{1.86}$$

In addition to the dimensionless numbers related to heat and mass transfer, the dimensionless quantities ν_1 and ν_2 are also contained in the above solutions. The role of ν_j, contrary to other dimensionless numbers, is strictly mathematical and has no physical significance. In the general case, i.e. equation (1.5), ν_j is a function of the Fedorov and Lu numbers:

$$\nu_j = \nu_j\,(\text{Fe, Lu}).$$

The values of ν_j are given in Table 15 for various Lu and Fedorov numbers. This table shows that ν_1 changes insignificantly with Lu and Fe. With increasing Lu and Fe numbers, its value decreases. At Lu = 0, ν_1 is equal to unity, and with Lu > 1.0, it decreases and tends to zero. With increasing Fe the ν_2 increases insignificantly, and with increasing Lu it decreases along an approximately logarithmic curve.

c) Infinite Plate. Parabolic Initial Conditions

$$\left[\; \Gamma = 0,\; \xi = x,\; f(x) = t(x, 0) = t_{\text{ц}} + \left(\frac{x}{R}\right)^2 (t_n - t_{\text{ц}}),\right.$$

$$\left. \varphi(x) = \theta\,(x, 0) = \theta_{\text{ц}} - \left(\frac{x}{R}\right)^2 (\theta_{\text{ц}} - \theta_n)\;\right].$$

For the solution of this problem we apply the Laplace transformations to equations (1) and (II):

$$a_q\, t''_L(x, s) - s\, t_L(x, s) +$$

$$+ \frac{\varepsilon\rho}{c}\; s\,\theta_L\,(x, s) - \frac{\varepsilon\rho}{c}\theta(x, 0) - t(x, 0) = 0, \tag{1.87}$$

$$a_m \theta''_L(x, s) - s\;\theta\,(x, s) + a_m\,\delta\, t''_L(x, s) + \theta\,(x, 0) = 0. \tag{1.88}$$

The values of $\theta_L(x, s)$, $\theta''_L(x, s)$, are obtained from equation (1.87) and substituted into equation (1.88). Following transformation we get an

Table 15. Values of $\nu_j = \nu_j$ (Lu, Fe) $j = 1,2$

Lu \ Fe	ν_j	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
0	ν_1	1	1	1	1	1	1	1	1	1	1	1
	ν_2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0,01	ν_1	1	0,9995	0,9990	0,9985	0,9980	0,9975	0,9970	0,9965	0,9960	0,9955	0,9950
	ν_2	10,0000	10,0050	10,0100	10,0150	10,0201	10,0251	10,0301	10,0352	10,0403	10,0453	10,0503
0,02	ν_1	1	0,9990	0,9980	0,9969	0,9960	0,9949	0,9939	0,9929	0,9919	0,9909	0,9899
	ν_2	7,0711	7,0782	7,0854	7,0926	7,0998	7,1070	7,1141	7,1213	7,1285	7,1356	7,1428
0,04	ν_1	1	0,9979	0,9959	0,9938	0,9918	0,9837	0,9877	0,9858	0,9838	0,9818	0,9798
	ν_2	5,0000	5,0104	5,0208	5,0311	5,0414	5,0517	5,0620	5,0722	5,0825	0,0927	5,1029
0,06	ν_1	1	0,9968	0,9937	0,9906	0,9875	0,9845	0,9815	0,9784	0,9754	0,9725	0,9696
	ν_2	4,0825	4,0955	4,1084	4,1213	4,1341	4,1469	4,1597	4,1724	4,1850	4,1977	4,2102
0,08	ν_1	1	0,9957	0,9914	0,9873	0,9831	0,9791	0,9750	0,9710	0,9671	0,9632	0,9595
	ν_2	3,5355	3,5508	3,5660	3,5812	3,5963	3,6113	3,6262	3,6410	3,6557	3,6704	2,6850
0,10	ν_1	1	0,9945	0,9891	0,9838	0,9786	0,9735	0,9684	0,9635	0,9587	0,9539	0,9492
	ν_2	3,1623	3,1798	3,1971	3,2143	3,2314	3,2484	3,2652	3,2820	3,2986	3,3151	3,3315
0,20	ν_1	1	0,9878	0,9762	0,9651	0,9544	0,9440	0,9344	0,9250	0,9158	0,9071	0,8986
	ν_2	2,2361	2,2636	2,2907	2,3170	2,3428	2,3681	2,3931	2,4175	2,4415	2,4652	2,4884
0,30	ν_1	1	0,9796	0,9609	0,9437	0,9278	0,9129	0,8989	0,8858	0,8733	0,8615	0,8503
	ν_2	1,8257	1,864	1,9000	1,9346	1,968	2,0000	2,031	2,0612	2,0906	2,1192	2,1472
0,40	ν_1	1	0,9694	0,9431	0,9200	0,8993	0,8803	0,8632	0,8473	0,8325	0,8187	0,8057
	ν_2	1,5811	1,6310	1,6764	1,7185	1,7582	1,7953	1,8316	1,8660	1,8992	1,9312	1,9623
0,60	ν_1	1	0,9415	0,9004	0,8677	0,8402	0,8167	0,7955	0,7766	0,7594	0,7437	0,7292
	ν_2	1,2910	1,3712	1,4339	1,4879	1,5365	1,5811	1,6229	1,6624	1,7000	1,7360	1,7706
0,80	ν_1	1	0,9020	0,8511	0,8137	0,7835	0,7580	0,7358	0,7162	0,6985	0,6824	0,6676
	ν_2	1,1180	1,2395	1,3136	1,3740	1,4269	1,4750	1,5193	1,5611	1,6007	1,6384	1,6746
1,00	ν_1	1	0,8543	0,8011	0,7630	0,7326	0,7071	0,6850	0,6657	0,6482	0,6324	0,6180
	ν_2	1,0000	1,1705	1,2483	1,3107	1,3650	1,4142	1,4597	1,5023	0,5427	1,5811	1,6181

Table 15, continued

Fe Lu	ν_j	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
0	ν_1	1	1	1	1	1	1	1	1	1	1	1
	ν_2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
2,00	ν_1	1	0,6526	0,6150	0,5858	0,5617	0,5409	0,5233	0,5073	0,4931	0,4801	0,4682
	ν_2	0,7071	1,0836	1,1.96	1,2070	1,2587	1,3067	1,3513	1,3938	1,4342	1,4730	1,5102
4,00	τ_1	1	0,4706	0,4472	0,4276	0,4108	0,3960	0,3831	0,3714	0,3609	0,3512	0,3423
	ν_2	0,5000	1,0623	1,1180	1,1693	1,2171	1,2622	1,3050	1,3461	1,3856	1,4236	1,4604
6,00	ν_1	1	0,3863	0,3680	0,3523	0,3389	0,3267	0,3162	0,3066	0,2979	0,2900	0,2826
	ν_2	0,4083	1,0571	1,1097	1,1587	1,2049	1,2487	1,2910	1,3314	1,3704	1,4080	1,4446
8,00	ν_1	1	0,3352	0,3198	0,3062	0,2949	0,2846	0,2753	0.2670	0,2594	0,2525	0,2461
	ν_2	0,3536	1,0548	1,1058	1,1544	1,1992	1,2524	1,2842	1,3243	1,3629	1,4004	1,4368
10,0	ν_1	1	0,3002	0,8865	0,2748	0,2645	0,2556	0,2470	0,2396	0,2328	0,2265	0,2208
	ν_2	0,3162	1,0535	1,1036	1,1509	1,1958	1,2388	1,2802	1,3201	1,3586	1,3960	1,4322
20,0	ν_1	1	0,2127	0,2034	0,1952	0,1880	0,1817	0,1758	0,1704	0,1657	0,1612	0,1572
	ν_2	0,2236	1,0510	1,0994	1,1454	1,1894	1,2318	1,2724	1,3118	1,3500	1,3870	1,4231
30,0	ν_1	1	0,1737	0,1661	0,1595	0,1538	0,1488	0,1437	0,1394	0,1355	0,1319	0,1285
	ν_2	0,1825	1,0503	1,0981	1,1436	1,1837	1,2293	1,2629	1,3091	1,3472	1,3841	1,4201
40,0	ν_1	1	0,1506	0,1439	0,1384	0,1332	0,1285	0,1247	0,1208	0,1174	0,1145	0,1114
	ν_2	0,1581	1,0499	1,0974	1,1427	1,1863	1,2282	1,2686	1,3078	1,3458	1,3827	1,4187
50,0	ν_1	1	0,1347	0,1288	0,1239	0,1194	0,1153	0,1116	0,1082	0,1051	0,1022	0,0998
	ν_2	0,1414	1,0497	1,0970	1,1422	1,1857	1,2275	1,2679	1,3070	1,3450	1,3819	1,4177
60,0	ν_1	1	0,1230	0,1179	0,1131	0,1091	0,1054	0,1020	0,0990	0,0962	0,0935	0,0911
	ν_2	0,1292	1,0495	1,0967	1,1419	1,1852	1,2270	1,2674	1,3065	1,3444	1,3812	1,4172
80,0	ν_1	1	0,1066	0,1020	0,0980	0,0913	0,0909	0,0883	0,0854	0,0831	0,0810	0,0791
	ν_2	0,1118	1,0493	1,0964	1,1415	1,1847	1,2265	1,2668	1,3058	1,3437	1,3805	1,4164
100	ν_1	1	0,0954	0,0911	0,0875	0,0843	0,0817	0,0791	0,0765	0,0745	0,0725	0,0707
	ν_2	0,1000	1,0492	1,0962	1,1412	1,1845	1,2261	2,2663	1,3054	1,3433	1,3801	0,4160

inhomogeneous ordinary differential equation

$$t_L^{\mathrm{IV}}(x, s) - \frac{s}{a_q} \Lambda t_L''(x, s) + \frac{s^2}{a_q a_m} t_L(x, s) + \\ + \frac{1}{2} w x^2 + \varkappa = 0, \tag{1.89}$$

where

$$\Lambda = 1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}},$$

$$w = 2 \frac{t_{\text{ц}} - t_n}{a_q R^2},$$

$$\varkappa = \frac{s}{a_q a_m} t_{\text{ц}} + w - 2 \frac{\varepsilon \rho}{c} \, \frac{\theta_{\text{ц}} - \theta_n}{a_q R^2}.$$

The general integral of equation (1.89) is found by another Laplace transform with respect to x. Here we have

$$t_L(x, s) = \frac{t_{\text{ц}} - (t_{\text{ц}} - t_n)\left(\frac{x^2}{R^2}\right)}{s} + \frac{2 a_m \omega}{s^2} + C_1 e^{\sqrt{\frac{s}{a_q}} \nu_1 x} + \\ + C_2 e^{\sqrt{\frac{s}{a_q}} \nu_2 x} + C_3 e^{-\sqrt{\frac{s}{a_q}} \nu_1 x} + C_4 e^{-\sqrt{\frac{s}{a_q}} \nu_2 x}, \tag{1.90}$$

where

$$\omega = \frac{t_{\text{ц}} - t_n}{R^2} - \frac{\varepsilon \rho}{c} \, \frac{\theta_{\text{ц}} - \theta_n}{R^2} - \frac{t_{\text{ц}} - t_n}{R^2} \Lambda, \tag{1.91}$$

ν_j is determined by equation (1.5), and
C_k are constants ($k = 1, 2, 3, 4$).

The solution for the transform of the mass potential, $\theta_L(x, s)$ is obtained by substituting $t_L(x, s)$ and $t_L''(x, s)$ into equation (1.87), i.e.:

$$\theta_L(x, s) = \frac{\theta_{\text{ц}} - (\theta_{\text{ц}} - \theta_n) \frac{x^2}{R^2}}{s} + \frac{a_q c}{\varepsilon \rho} \, \frac{2\left(\frac{t_{\text{ц}} - t_n}{R^2} + \omega \, \mathrm{Lu}\right)}{s^2} + \\ + \frac{c}{\varepsilon \rho} \left[C_1 (1 - \nu_1^2) e^{\sqrt{\frac{s}{a_q}} \nu_1 x} + C_2 (1 - \nu_2^2) e^{\sqrt{\frac{s}{a_q}} \nu_2 x} + \right.$$

$$+ C_3(1 - \nu_1^2)\, e^{-\sqrt{\frac{s}{a_q}}\,\nu_1 x} + C_4(1 - \nu_2^2)\, e^{-\sqrt{\frac{s}{a_q}}\,\nu_2 x}\Bigg]. \qquad (1.92)$$

Since the condition of symmetry (V) shows that $C_1 = C_3$ and $C_2 = C_4$, equations (1.90) and (1.92) can be rewritten as

$$t_L(x, s) = \frac{t_{\text{ц}} - (t_{\text{ц}} - t_n)\left(\frac{x^2}{R^2}\right)}{s} + \frac{2 a_m \omega}{s^2} +$$

$$+ B_1 \operatorname{ch} \sqrt{\frac{s}{a_q}}\,\nu_1 x + B_2 \operatorname{ch} \sqrt{\frac{s}{a_q}}\,\nu_2 x, \qquad (1.93)$$

$$\theta_E(x, s) = \frac{\theta_{\text{ц}} - (\theta_{\text{ц}} - \theta_n)\frac{x^2}{R^2}}{s} + \frac{2\frac{a_q c}{\varepsilon\rho}\left(\frac{t_{\text{ц}} - t_n}{R^2} + \omega \mathrm{Lu}\right)}{s^2} +$$

$$+ B_1 \frac{c}{\varepsilon\rho}(1 - \nu_1^2) \operatorname{ch} \sqrt{\frac{s}{a_q}}\,\nu_1 x +$$

$$+ B_2 \frac{c}{\varepsilon\rho}(1 - \nu_2^2) \operatorname{ch} \sqrt{\frac{s}{a_q}}\,\nu_2 x. \qquad (1.94)$$

B_1 and B_2 are new constants with respect to x.

Satisfying equations (1.93) and (1.94) with boundary conditions (1.8) and (1.9) we can determine B_j from

$$\left.\begin{aligned} B_1 Q_1 + B_2 Q_2 - \frac{\Omega_1}{s} - \frac{\Omega_2}{s^2} &= 0 \\ B_1 P_1 + B_2 P_2 - \frac{\Omega_3}{s} - \frac{\Omega_4}{s^2} &= 0 \end{aligned}\right\} \qquad (1.95)$$

where

$$\Omega_1 = \frac{2}{\mathrm{Bi}_q}(t_{\text{ц}} - t_n) + (t_c - t_n) - \varepsilon K_1 \frac{\rho}{c}(\theta_n - \theta_p), \qquad (1.96)$$

$$\Omega_2 = -2a_q\left[K_1\left(\frac{t_{\text{ц}} - t_n}{R^2} + \omega \mathrm{Lu}\right) + \omega \mathrm{Lu}\right], \qquad (1.97)$$

$$\Omega_3 = 2\frac{\mathrm{Fe}}{\mathrm{Bi}_m}\left[\frac{1}{\delta}(\theta_{\text{ц}} - \theta_n) + (t_{\text{ц}} - t_n) - \frac{\mathrm{Bi}_m}{2\delta}(\theta_n - \theta_p)\right], \qquad (1.98)$$

$$\Omega_4 = -2\,a_q\left(\frac{t_{\text{ц}} - t_n}{R^2} + \omega\,\mathrm{Lu}\right), \tag{1.99}$$

Q_j and P_j are determined by equations (1.13) and (1.14), respectively. The solution of equation (1.95) yields

$$B_1 = -\left[\frac{\Omega_1 P_2 - \Omega_3 Q_2}{s\,(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2 P_2 - \Omega_4 Q_2}{s^2\,(Q_2 P_1 - Q_1 P_2)}\right],$$

$$B_2 = \frac{\Omega_1 P_1 - \Omega_3 Q_1}{s\,(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2 P_1 - \Omega_4 Q_1}{s^2\,(Q_2 P_1 - Q_1 P_2)}.$$

Substituting B_1 and B_2 into equations (1.93) and (1.94) we obtain the following general solutions for the distribution of transfer potentials:

$$\begin{aligned} t_L(x, s) &= \frac{t_{\text{ц}} - (t_{\text{ц}} - t_n)\,\dfrac{x^2}{R^2}}{s} + \frac{2a_m\,\omega}{s^2} + \\ &+ \left\{\left[\frac{\Omega_1 P_1 - \Omega_3 Q_1}{s\,(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2 P_1 - \Omega_4 Q_1}{s^2\,(Q_2 P_1 - Q_1 P_2)}\right]\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 x - \right. \\ &- \left.\left[\frac{\Omega_1 P_2 - \Omega_3 Q_2}{s\,(Q_2 P_1 - Q_1 P_2)} + \frac{\Omega_2 P_2 - \Omega_4 Q_2}{s^2\,(Q_2 P_1 - Q_1 P_2)}\right]\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x\right\}, \end{aligned} \tag{1.100}$$

$$\begin{aligned} \theta_L(x, s) &= \frac{\theta_{\text{ц}} - (\theta_{\text{ц}} - \theta_n)\,\dfrac{x^2}{R^2}}{s} + \frac{2\,\dfrac{a_q c}{\varepsilon\rho}\left(\dfrac{t_{\text{ц}} - t_n}{R^2} + \omega\,\mathrm{Lu}\right)}{s^2} + \\ &+ \left\{\left[\frac{\Omega_1 P_1 - \Omega_3 Q_1}{s\,(Q_2 P_1 - Q_1 P_2)} + \right.\right. \\ &+ \left.\frac{\Omega_2 P_1 - \Omega_4 Q_1}{s^2\,(Q_2 P_1 - Q_1 P_2)}\right]\frac{c}{\varepsilon\rho}\,(1 - \nu_2^2)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x - \\ &- \left[\frac{\Omega_1 P_2 - \Omega_3 Q_2}{s\,(Q_2 P_1 - Q_1 P_2)} + \right. \\ &+ \left.\left.\frac{\Omega_2 P_2 - \Omega_4 Q_2}{s^2\,(Q_2 P_1 - Q_1 P_2)}\right]\frac{c}{\varepsilon\rho}\,(1 - \nu_1^2)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x\right\}. \end{aligned} \tag{1.101}$$

In equations (1.100) and (1.101) we have two groups of terms which have to be inverted:

$$\frac{\Omega_1 P_j - \Omega_3 Q_j}{s\,(Q_2P_1 - Q_1P_2)} \operatorname{ch} \sqrt{\frac{s}{a_q}}\,\underline{\nu}\, x$$

and

$$\frac{\Omega_2 P_j - \Omega_4 Q_j}{s^2\,(Q_2P_1 - Q_1P_2)} \operatorname{ch} \sqrt{\frac{s}{a_q}}\,\underline{\nu}\, x\,.$$

Here, $\underline{\nu} = \nu_{j-(-1)^j}$ $(j = 1, 2)$.

We shall now consider the transform of the first group:

$$L^{-1}\left[\frac{(\Omega_1 P_j - \Omega_3 Q_j)\operatorname{ch}\sqrt{\frac{s}{a_q}}\,\underline{\nu} x}{s\,(Q_2P_1 - Q_1P_2)}\right] =$$

$$= L^{-1}\left[\frac{\Phi_{01}(s)}{\psi_{01}(s)}\right]. \tag{1.102}$$

Since the conditions of the expansion theorem are fulfilled, and the roots of the denominator $\psi_{01}(s) = 0$ are

$s = 0$ (zeroth root), and

$s = s_n$, where $s_n = -\dfrac{a_q \mu_n^2}{R^2}$,

the theorem of expansion may be written in the following manner:

$$L^{-1}\left[\frac{\Phi_{01}(s)}{\psi_{01}(s)}\right] = \frac{\Phi_{01}(0)}{\psi'_{01}(0)} + \sum_{n=1}^{\infty} \frac{\Phi_{01}(s_n)}{\psi'_{01}(s_n)} \exp(s_n \tau). \tag{1.102'}$$

For $\Phi_{01}(0)$, $\psi'_{01}(0)$, $\Phi_{01}(s_n)$, $\psi'_{01}(s_n)$ the calculation yields:

$$\Phi_{01}(0) = (\Omega_1 - K_1\Omega_3)(1 - \nu_j^2) - \Omega_3\,,$$

$$\psi'_{01}(0) = \nu_2^2 - \nu_1^2\,,$$

$$\Phi_{01}(s_n) = (\Omega_1 P_{nj} - \Omega_3 Q_{nj}) \cos \mu_n \underline{\nu} \frac{x}{R}\,,$$

$$\psi'_{01}(s_n) = \frac{\mu_n}{2}\psi_n\,,$$

where μ_n, Q_{nj}, P_{nj} and ψ_n are described by equations (1.20) to (1.23), respectively.

Substituting the expressions found in equations (1.102') and (1.102) we get the inversion of the transform of the first group of terms:

$$L^{-1}\left[\frac{(\Omega_1 P_j - \Omega_3 Q_j)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu x}{s\,(Q_2P_1 - Q_1P_2)}\right] =$$

$$= \frac{(\Omega_1 - K_1\,\Omega_3)\,(1 - \nu_j^2) - \Omega_3}{\nu_2^2 - \nu_1^2} +$$

$$+ \sum_{n=1}^{\infty} 2\,\frac{(\Omega_1 P_{nj} - \Omega_3 Q_{nj})\cos \nu\mu_n \frac{x}{R}}{\mu_n\,\psi_n}\exp(-\mu_n^2\,\mathrm{Fo}). \qquad (1.103)$$

Let us consider the inversion of the transform of the second group

$$L^{-1}\left[\frac{(\Omega_2 P_j - \Omega_4 Q_j)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu x}{s^2\,(Q_2P_1 - Q_1P_2)}\right] =$$

$$= L^{-1}\left[\frac{\Phi_{02}(s)}{\psi_{02}(s)}\right] = L^{-1}\left[\frac{\Phi_{02}(s)}{s^2\,\varphi(s)}\right].$$

This case is different from the preceding in that there is a double zeroth root. In order to obtain this root we utilize the expansion theorem for multiple roots:

$$L^{-1}\left[\frac{\Phi_{02}(0)}{\psi_{02}(0)}\right] = \lim_{s\to 0}\left[\frac{d}{ds}\left(\frac{\Phi_{02}(s)s^2}{\psi_{02}(s)}\,e^{s\tau}\right)\right] =$$

$$= \lim_{s\to 0}\left[\frac{d}{ds}\left(\frac{\Phi_{02}(s)}{\varphi(s)}\,e^{s\tau}\right)\right] =$$

$$= \lim_{s\to 0}\left[\frac{\tau\,e^{s\tau}\,\Phi_{02}(s)}{\varphi(s)} + \frac{e^{s\tau}\,\Phi'_{02}(s)}{\varphi(s)} - \right.$$

$$\left. - \frac{e^{s\tau}\,\Phi_{02}(s)\,\varphi'(s)}{[\varphi(s)]^2}\right].$$

The terms appearing in the expansion theorem have the following values:

$$\varphi(0) = \nu_2^2 - \nu_1^2,$$

$$\varphi'(0) = \frac{R^2}{2a_q}\left(1 + \mathrm{Fe} + \frac{2}{\mathrm{Bi}_q} + \frac{1}{\mathrm{Lu}} + \right.$$

$$\left. + \frac{2}{\mathrm{Bi}_m \mathrm{Lu}}\right)(\nu_2^2 - \nu_1^2),$$

$$\Phi_{02}(0) = (\Omega_2 - K_1\Omega_4)(1 - \nu_j^2) - \Omega_4,$$

$$\Phi'_{02}(0) = [(\Omega_2 - K_1\Omega_4)(1 - \nu_j^2) - \Omega_4]\,\frac{\nu_j^2 R^2 + \underline{\nu}^2 x^2}{2a_q} +$$

$$+ \left[\Omega_2\frac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m} - \Omega_4\frac{1}{\mathrm{Bi}_q}\right]\frac{\nu_j^2 R^2}{a_q},$$

$$\psi'_{02}(s_n) = -\frac{a_q}{2R^2}\,\mu_n^3\,\psi_n,$$

$$\Phi_{02}(s_n) = (\Omega_2 P_{nj} - \Omega_4 Q_{nj})\cos\underline{\nu}\,\mu_n\,x/R.$$

Therefore, the inverted transform of the second group can be described as follows:

$$L^{-1}\left[\frac{(\Omega_2 P_j - \Omega_4 Q_j)\,\mathrm{ch}\,\sqrt{\dfrac{s}{a_q}}\,\underline{\nu}\,x}{s^2(Q_2P_1 - Q_1P_2)}\right] =$$

$$= \tau\,\frac{(\Omega_2 - K_1\Omega_4)(1 - \nu_j^2) - \Omega_4}{\nu_2^2 - \nu_1^2} +$$

$$+ \frac{[(\Omega_2 - K_1\Omega_4)(1 - \nu_j^2) - \Omega_4]\dfrac{\nu_j^2 R^2 + \underline{\nu}^2 x^2}{2a_q}}{\nu_2^2 - \nu_1^2} +$$

$$+ \frac{\left[\Omega_2\dfrac{(1 - \nu_j^2) + \mathrm{Fe}}{\mathrm{Bi}_m} - \Omega_4\dfrac{1}{\mathrm{Bi}_q}\right]\dfrac{\nu_j^2 R^2}{a_q}}{\nu_2^2 - \nu_1^2} -$$

$$-\frac{\frac{R^2}{2a_q}\left(1+\mathrm{Fe}+\frac{2}{\mathrm{Bi}_q}+\frac{1}{\mathrm{Lu}}+\frac{2}{\mathrm{Bi}_m\mathrm{Lu}}\right)}{\nu_2^2-\nu_1^2}\times$$

$$\times\ [(\Omega_2-K_1\Omega_4)(1-\nu_j^2)-\Omega_4]\ -$$

$$-\sum_{n=1}^{\infty}\frac{2R^2}{a_q}\ \frac{(\Omega_2 P_{nj}-\Omega_4 Q_{nj})\cos\nu_j\mu_n\frac{x}{R}}{\mu_n^3\psi_n}\times$$

$$\times\exp(-\mu_n^2\mathrm{Fo}). \tag{1.104}$$

The inversion for the transfer potential fields are obtained by applying the expansion theorem to equations (1.100) and (1.101); here we consider the solution found for the inverted transforms (1.103) and (1.104). Solving for Λ, ω, Ω_k ($k=1, 2, 3, 4$) by using equations (1.91) and (1.96) to (1.99), simplifying, and reducing the terms to dimensionless parameters, we finally obtain

$$T_1=\frac{t(x,\tau)-t_n}{t_c-t_n}=1-\sum_{n=1}^{\infty}(C_{n2}\cos\nu_1\mu_n x/R\ -$$

$$-\ C_{n1}\cos\nu_2\mu_n x/R)\exp(-\mu_n^2\mathrm{Fo}), \tag{1.105}$$

$$\Theta_1=\frac{\theta_n-\theta(x,\tau)}{\theta_n-\theta_p}=1-\sum_{n=1}^{\infty}[C_{n1}^*(1-\nu_2^2)\cos\nu_2\mu_n x/R\ -$$

$$-C_{n2}^*(1-\nu_1^2)\cos\nu_1\mu_n x/R]\exp(-\mu_n^2\mathrm{Fo}), \tag{1.106}$$

$$C_{nj}=2\frac{G_1P_{nj}-G_2Q_{nj}}{\mu_n\psi_n},$$

$$C^*{}_{nj}=\frac{C_{nj}}{\varepsilon\mathrm{Ko}}, \tag{1.107}$$

$$G_1=1-2W\left(\frac{1}{\mathrm{Bi}_q}-\frac{1}{\mu_n^2}\right)-$$

$$-\varepsilon\mathrm{Ko}K_1-2\varepsilon\mathrm{Ko}K_2\frac{(K_1+1)\mathrm{Lu}}{\mu_n^2}, \tag{1.108}$$

$$G_2=2\varepsilon\mathrm{Ko}K_2\left(\frac{1}{\mathrm{Bi}_m}-\frac{\mathrm{Lu}}{\mu_n^2}\right)-\varepsilon\mathrm{Ko}. \tag{1.109}$$

In problems with a parabolic initial condition, the dimensionless numbers Lu, Bi_q, Bi_m, Ko, Pn, ε, K_1, and Fe preserve their form when t_0 and θ_0 are substituted by t_n and θ_n, respectively.

The characteristic equation in the parabolic problem is identical with the characteristic equations for a plate with uniform initial conditions. Therefore, the values of μ_n given in Tables 1-4 are also roots of this problem. Comparison of the starting values of Fo for parabolic initial conditions (Table 6) with analogous values for constant initial conditions, does not reveal any significant differences. It can therefore be concluded that the initial conditions do not affect the simplified solutions of internal heat and mass transfer equations. The heat and mass transfer coefficients for a plate calculated from equations (1.107), as functions of W and V are given in Tables 16 and 17, and also in Figures 21 and 22.

The average values of dimensionless temperature and mass transfer potential with parabolic initial conditions can be expressed as follows:

$$\overline{T}_1 = \frac{\overline{t}(\tau) - t_n}{t_c - t_n} = 1 - \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2 \mathrm{Fo}), \qquad (1.110)$$

$$\overline{\Theta}_1 = \frac{\theta_n - \overline{\theta}(\tau)}{\theta_n - \theta_p} = 1 - \sum_{n=1}^{\infty} D_n^* \exp(-\mu_n^2 \mathrm{Fo}), \qquad (1.111)$$

D_n and D_n^* are given by equations (1.38) and (1.39), as previously, with new values of C_{nj} and C_{nj}^*.

For a plate, the dimensionless rate of heating and drying can be obtained by differentiating equations (1.105) and (1.106), or (1.110) and (1.111), with respect to the dimensionless time, i.e.,

$$\frac{dT_1}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} (C_{n2} \cos \nu_1 \mu_n x/R -$$

$$- C_{n1} \cos \nu_2 \mu_n x/R) \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}),$$

$$\frac{d\overline{T}_1}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}),$$

$$\frac{d\Theta_1}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} [C_{n1}^* (1 - \nu_2^2) \cos \nu_2 \mu_n x/R -$$

$$- C_{n2}^* (1 - \nu_1^2) \cos \nu_1 \mu_n x/R] \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}),$$

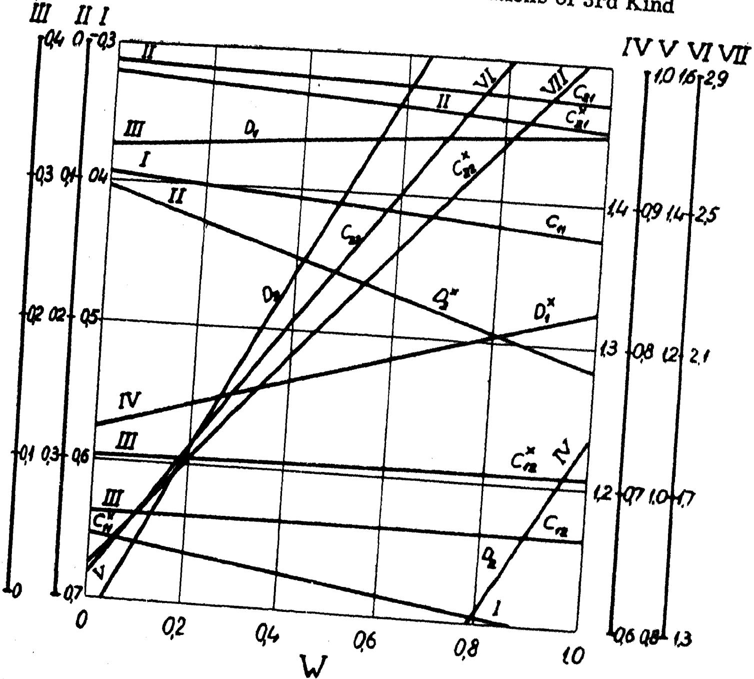

Fig. 21. Heat and mass transfer coefficients for an infinite plate as a function of W.

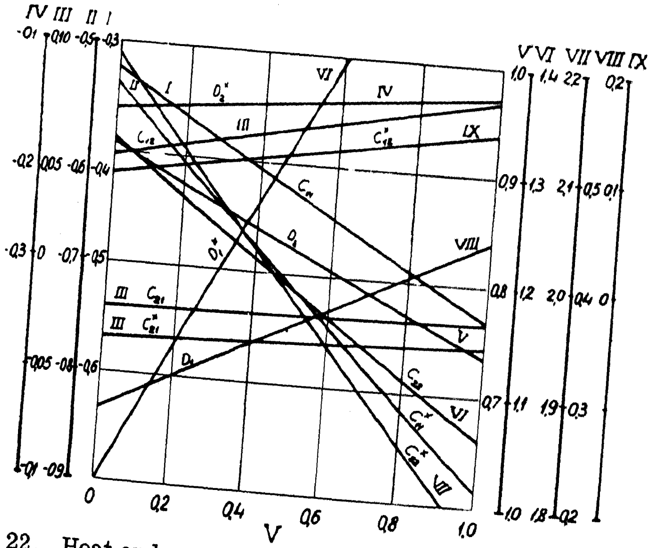

Fig. 22. Heat and mass transfer coefficients as a function of V for an infinite plate.

Table 16. Coefficients in the solution of heat and mass transfer equations as a function of W (infinite plate with parabolic initial conditions).

$\mathrm{Lu} = 0{,}3$, $\varepsilon = 0{,}5$, $\mathrm{Ko} = 1{,}2$, $\mathrm{Pn} = 0{,}5$, $\mathrm{Bi}_m = 10$, $\mathrm{Bi}_q = 10$, $V = 0{,}4$

W	C_{n1}		C_{n2}		C^*_{n1}		C^*_{n2}		D_n		D^*_n	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—0,3918	—0,01300	0,06262	0,8372	—0,6531	—0,02167	0,1044	1,396	0,3274	0,5802	1,223	—0,1015
0,2	—0,3982	—0,01585	0,06286	1,026	—0,6638	—0,02642	0,1048	1,710	0,3320	0,7111	1,243	—0,1243
0,4	—0,4046	—0,01868	0,06310	1,214	—0,6745	—0,03114	0,1052	2,024	0,3366	0,8414	1,263	—0,1472
0,6	—0,4110	—0,02154	0,06336	1,403	—0,6851	—0,03591	0,1056	2,339	0,3413	0,9724	1,283	—0,1701
0,8	—0,4174	—0,02438	0,06360	1,591	—0,6958	—0,04064	0,1060	2,652	0,3459	1,103	1,303	—0,1929
1,0	—0,4238	—0,02722	0,06384	1,779	—0,7065	—0,04538	0,1064	2,966	0,3505	1,233	1,324	—0,2157

Table 17. Coefficients in the solution of heat and mass transfer equations as a function of V (infinite plate with parabolic initial conditions).

$\mathrm{Lu} = 0{,}3$, $\varepsilon = 0{,}5$, $\mathrm{Ko} = 1{,}2$, $\mathrm{Pn} = 0{,}5$, $\mathrm{Bi}_m = 10$, $\mathrm{Bi}_q = 10$, $W = 0{,}4$

V	C_{n1}		C_{n2}		C^*_{n1}		C^*_{n2}		D_n		D^*_n	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—0,3208	—0,02020	0,04926	1,315	—0,5348	—0,03367	0,08212	2,192	0,2662	0,9114	1,002	—0,1594
0,2	—0,3624	—0,01944	0,05615	1,264	—0,6045	—0,03241	0,09362	2,107	0,3012	0,8761	1,133	—0,1532
0,4	—0,4046	—0,01868	0,06310	1,214	—0,6745	—0,03114	0,1052	2,024	0,3366	0,8414	1,263	—0,1472
0,6	—0,4466	—0,01794	0,07002	1,164	—0,7445	—0,02991	0,1167	1,940	0,3719	0,8068	1,394	—0,1411
0,8	—0,4882	—0,01718	0,07692	1,113	—0,8138	—0,02864	0,1282	1,855	0,4069	0,7714	1,524	—0,1349
1,0	—0,5300	—0,01641	0,08384	1,063	—0,8835	—0,02736	0,1398	1,772	0,4420	0,7368	1,655	—0,1288

$$\frac{d\bar{\Theta}_1}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n^* \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}).$$

d) Sphere. Parabolic Initial Conditions

$$\left[\Gamma=2,\ \xi = r,\ f(r) = t_{\text{ц}} + \left(\frac{r}{R}\right)^2 (t_n - t_{\text{ц}}),\ \varphi(r) = \theta_{\text{ц}} - \left(\frac{r}{R}\right)^2 (\theta_{\text{ц}} - \theta_n)\right].$$

The general solutions for heat and mass transfer potentials in terms of Laplace transforms can be obtained by the method presented in the previous problem. For the heat transfer potential:

$$\vartheta_L(r, s) = \frac{r\left[t_{\text{ц}} - (t_{\text{ц}} - t_n)\frac{r^2}{R^2}\right]}{s} + \frac{6\, r a_m \omega}{s^2} + C_1 e^{\sqrt{\frac{s}{a_q}}\nu_1 r} + C_2 e^{\sqrt{\frac{s}{a_q}}\nu_2 r} + C_3 e^{-\sqrt{\frac{s}{a_q}}\nu_1 r} + C_4 e^{-\sqrt{\frac{s}{a_q}}\nu_2 r}; \qquad (1.112)$$

for the mass transfer potential

$$u_L(r, s) = \frac{r\left[\theta_{\text{ц}} - (\theta_{\text{ц}} - \theta_n)\frac{r^2}{R^2}\right]}{s} + \frac{a_q c}{\varepsilon\rho}\,\frac{6r\left[\frac{t_{\text{ц}} - t_n}{R^2} + \omega\,\mathrm{Lu}\right]}{s^2} + \frac{c}{\varepsilon\rho}\left[C_1(1-\nu_1^2)e^{\sqrt{\frac{s}{a_q}}\nu_1 r} + C_2(1-\nu_2^2)e^{\sqrt{\frac{s}{a_q}}\nu_2 r} + C_3(1-\nu_1^2)e^{-\sqrt{\frac{s}{a_q}}\nu_1 r} + C_4(1-\nu_2^2)e^{-\sqrt{\frac{s}{a_q}}\nu_2 r}\right], \qquad (1.113)$$

where C_k (k = 1, 2, 3, 4) are constants with respect to r, ν_j, and ω which are described by equations (1.5) and (1.91).

We utilize the conditions of symmetry with equations (1.112) and (1.113). In the solution of problems with uniform initial conditions it has been shown that the conditions of symmetry (V) for a sphere can be represented by equations (1.51) and (1.52):

$$\vartheta_L(0, s) = 0, \quad u_L(0, s) = 0.$$

The application of these conditions to equations (1.112) and (1.113) gives

$$C_1 = -C_3, \; C_2 = -C_4.$$

Consequently, the solutions of transforms of the modified temperature $\vartheta_L(r, s)$ and mass transfer potential, or $u_L(r, s)$, are

$$\vartheta_L(r, s) = \frac{r\left[t_{ц} - (t_{ц} - t_{n})\frac{r^2}{R^2}\right]}{s} + \frac{6\, r a_m \omega}{s^2} + \\ + B_1 \operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r + B_2 \operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r, \tag{1.114}$$

$$u_L(r, s) = \frac{r\left[\theta_{ц} - (\theta_{ц} - \theta_{n})\frac{r^2}{R^2}\right]}{s} + \\ + \frac{6\, r \frac{a_q c}{\varepsilon\rho}\left[\frac{t_{ц} - t_{n}}{R^2} + \omega\,\mathrm{Lu}\right]}{s^2} + \\ + B_1 \frac{c}{\varepsilon\rho}(1 - \nu_1^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r + \\ + B_2 \frac{c}{\varepsilon\rho}(1 - \nu_2^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r. \tag{1.115}$$

B_1 and B_2 are new constants with respect to the coordinates, whose values are determined from boundary conditions (III) to (IV) or from equations (1.57) and (1.58). Substituting (1.114) and (1.115) into the boundary conditions and solving the resulting systems of equations for B_1 and B_2, yields

$$B_1 = -\left[\frac{\Omega_1 \tilde{P}_2 - \Omega_3 \tilde{Q}_2}{s(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} + \frac{\Omega_2 \tilde{P}_2 - \Omega_4 \tilde{Q}_2}{s^2(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)}\right],$$

$$B_2 = \frac{\Omega_1 \tilde{P}_1 - \Omega_3 \tilde{Q}_1}{s(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)} + \frac{\Omega_2 \tilde{P}_1 - \Omega_4 \tilde{Q}_1}{s^2(\tilde{Q}_2 \tilde{P}_1 - \tilde{Q}_1 \tilde{P}_2)}.$$

Here, $\widetilde{Q}_j$ and $\widetilde{P}_j$ are determined by equations (1.60) and (1.61); Ω_1 and Ω_3 are determined by equations (1.96) and (1.98), and

$$\Omega_2 = -6\, a_q \left[K_1 \left(\frac{t_ц - t_n}{R^2} + \omega\, \mathrm{Lu} \right) + \omega\, \mathrm{Lu} \right], \tag{1.116}$$

$$\Omega_4 = -6\, a_q \left[\frac{t_ц - t_n}{R^2} + \omega\, \mathrm{Lu} \right]. \tag{1.117}$$

Consequently, the final solution can be written as

$$\begin{aligned} \vartheta_L (r, s) = {} & \frac{r\left[t_ц - (t_ц - t_n)\dfrac{r^2}{R^2} \right]}{s} + \frac{6\, r a_m \omega}{s^2} + \\ & + \left\{ \left[\frac{\Omega_1 \widetilde{P}_1 - \Omega_3 \widetilde{Q}_1}{s(\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} + \frac{\Omega_2 \widetilde{P}_1 - \Omega_4 \widetilde{Q}_1}{s^2 (\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} \right] \mathrm{sh} \sqrt{\frac{s}{a_q}}\, \nu_2 r - \right. \\ & - \left[\frac{\Omega_1 \widetilde{P}_2 - \Omega_3 \widetilde{Q}_2}{s(\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} + \right. \\ & \left. \left. + \frac{\Omega_2 \widetilde{P}_2 - \Omega_4 \widetilde{Q}_2}{s^2 (\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} \right] \mathrm{sh} \sqrt{\frac{s}{a_q}}\, \nu_1 r \right\}, \end{aligned} \tag{1.118}$$

$$\begin{aligned} u_L (r, s) = {} & \frac{r\left[\theta_ц - (\theta_ц - \theta_n)\dfrac{r^2}{R^2} \right]}{s} + \\ & + \frac{c}{\varepsilon \rho} \left\{ \frac{6\, r a_q \left[\dfrac{t_ц - t_n}{R^2} + \omega\, \mathrm{Lu} \right]}{s^2} + \right. \\ & + \left[\frac{\Omega_1 \widetilde{P}_1 - \Omega_3 \widetilde{Q}_1}{s(\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} + \right. \\ & \left. + \frac{\Omega_2 \widetilde{P}_1 - \Omega_4 \widetilde{Q}_1}{s^2 (\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} \right] (1 - \nu_2^2)\, \mathrm{sh} \sqrt{\frac{s}{a_q}}\, \nu_2 r - \\ & - \left[\frac{\Omega_1 \widetilde{P}_2 - \Omega_3 \widetilde{Q}_2}{s(\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} + \right. \\ & \left. \left. + \frac{\Omega_2 \widetilde{P}_2 - \Omega_4 \widetilde{Q}_2}{s^2 (\widetilde{Q}_2 \widetilde{P}_1 - \widetilde{Q}_1 \widetilde{P}_2)} \right] (1 - \nu_1^2)\, \mathrm{sh} \sqrt{\frac{s}{a_q}}\, \nu_1 r \right\}. \end{aligned} \tag{1.119}$$

We shall now get two basic inversions of transforms, needed to obtain the inverse transformation of equations (1.118) and (1.119):

$$L^{-1}\left[\frac{(\Omega_1\tilde{P}_j-\Omega_3\tilde{Q}_j)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu r}{s(\tilde{Q}_2\tilde{P}_1-\tilde{Q}_1\tilde{P}_2)}\right]=L^{-1}\left[\frac{\Phi_1(s)}{\psi_1(s)}\right]. \qquad (1.120)$$

Because neither the numerator nor the denominator of equation (1.120) are generalized polynomials with respect to s, the expansion theorem can be applied only after a number of manipulations: the numerator and denominator are expanded in series around the zeroth root; both parts are divided by $\sqrt{\frac{s}{a_q}}\ \sqrt{\frac{s}{a_q}}$, ; the derivative $\psi_{01}(s)$ is taken with respect to s, and the limit of the expression is found as s tends to zero (see section 1b).

Thus, the theorem of expansion for expression (1.120) is

$$L^{-1}\left|\frac{\Phi_1(s)}{\psi_1(s)}\right|=\frac{\Phi_{01}(0)}{\psi_0'(0)}+\sum_{n=1}^{\infty}\frac{\Phi_1(s_n)}{\psi_1'(s_n)}\exp(s_n\tau), \quad (1.121)$$

where the first term corresponds to that for the zeroth root, $\psi_1(s)=s(\tilde{Q}_2\tilde{P}_1-\tilde{Q}_1\tilde{P}_2)=0$, and the second term, $\tilde{Q}_2\tilde{P}_1-\tilde{Q}_1\tilde{P}_2=0$; corresponds to the roots $s_n=-\frac{a_q\mu_n^2}{R^2}$ with μ_n determined by the roots of equation (1.65).

The solution of (1.120) or (1.121) yields

$$L^{-1}\left[\frac{(\Omega_1\tilde{P}_j-\Omega_3\tilde{Q}_j)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu r}{s(\tilde{Q}_2\tilde{P}_1-\tilde{Q}_1\tilde{P}_2)}\right]=$$

$$=\frac{(\Omega_1-K_1\Omega_3)(1-\nu_j^2)-\Omega_3}{\nu_2^2-\nu_j^2}\,r-$$

$$-\sum_{n=1}^{\infty}2\,\frac{\Omega_1P_{nj}-\Omega_3Q_{nj}}{\mu_n\,\psi_n}\,R\sin\nu\mu_n\frac{r}{R}\exp(-\mu_n^2\,\mathrm{Fo}). \qquad (1.122)$$

The second basic inversion will be obtained in the same manner as in the plate problem. In this problem, however, additional manipulations must be performed after the polynomials have been reduced to conditions satisfying the generalized expansion theorem

$$
\begin{aligned}
L^{-1}&\left[\frac{(\Omega_2\widetilde{P}_j-\Omega_4\widetilde{Q}_j)\,\mathrm{sh}\sqrt{\dfrac{s}{a_q}}\,\underline{\nu}\,r}{s^2(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\right]=\\
&=\frac{r\,[(\Omega_2-K_1\Omega_4)(1-\nu_j^2)-\Omega_4]\left[\tau+\dfrac{\underline{r}^2\nu^2+R^2\nu_j^2}{6\,a_q}-\right.}{\nu_2^2-\nu_1^2}\\
&\quad\frac{\left.-\left(1+\dfrac{1}{\mathrm{Lu}}+\mathrm{Fe}+\dfrac{2}{\mathrm{Bi}_q}+\dfrac{2}{\mathrm{Bi}_m}\,\dfrac{1}{\mathrm{Lu}}\right)\dfrac{R^2}{6\,a_q}\right]}{\nu_2^2-\nu_1^2}+\\
&\quad+\frac{\left[\Omega_2\dfrac{(1-\nu_j^2)+\mathrm{Fe}}{\mathrm{Bi}_m}-\Omega_4\dfrac{1}{\mathrm{Bi}_q}\right]\dfrac{2\,\nu_j^2R^2}{6\,a_q}}{\nu_2^2-\nu_1^2}\,r+\\
&\quad+\sum_{n=1}^{\infty}2\,\frac{R^2}{a_q}\,\frac{\Omega_2P_{nj}-\Omega_4Q_{nj}}{\mu_n^3\,\psi_n}\,R\sin\underline{\nu}\,\mu_n\,\frac{r}{R}\,\exp(-\mu_n^2\,\mathrm{Fo}).
\end{aligned}
\tag{1.123}
$$

The quantities, ψ_n, A_{nj}, and B_{nj} are described by equations (1.70) and (1.72).

Applying the expansion theorem to equations (1.118) and (1.119), and using equations (1.122) and (1.123), we get the following expressions for the temperature and mass potential fields:

$$
\begin{aligned}
T_1=\frac{t(r,\tau)-t_n}{t_c-t_n}=1-\sum_{n=1}^{\infty}&\left[C_{n1}\,\frac{\sin\,\nu_2\mu_n r/R}{r/R}-\right.\\
&\left.-C_{n2}\,\frac{\sin\nu_1\mu_n r/R}{r/R}\right]\exp(-\mu_n^2\,\mathrm{Fo}).
\end{aligned}
\tag{1.124}
$$

$$
\begin{aligned}
\Theta_1=\frac{\theta_n-\theta(r,\tau)}{\theta_n-\theta_p}=1-\sum_{n=1}^{\infty}&\left[C_{n2}^{*}(1-\nu_1^2)\,\frac{\sin\nu_1\mu_n r/R}{r/R}-\right.\\
&\left.-C_{n1}^{*}(1-\nu_2^2)\,\frac{\sin\nu_2\mu_n r/R}{r/R}\right]\exp(-\mu_n^2\,\mathrm{Fo}),
\end{aligned}
\tag{1.125}
$$

$$
C_{nj}=2\,\frac{G_1P_{nj}-G_2Q_{nj}}{\mu_n\,\psi_n},\quad C_{nj}^{*}=\frac{1}{\varepsilon\,\mathrm{Ko}}\,C_{nj},
\tag{1.126}
$$

$$G_1 = 1 - 2\,W\left(\frac{1}{\mathrm{Bi}_q} - \frac{3}{\mu_n^2}\right) -$$

$$-\,\varepsilon\,\mathrm{Ko}\,K_1 - 6\,\varepsilon\,\mathrm{Ko}K_2\,\frac{(K_1+1)\,\mathrm{Lu}}{\mu_n^2}, \tag{1.127}$$

$$G_2 = 2\,\varepsilon\,\mathrm{Ko}\,K_2\left(\frac{1}{\mathrm{Bi}_m} - \frac{3\,\mathrm{Lu}}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko}. \tag{1.128}$$

In this case, the expressions for the dimensionless temperature and mass transfer potentials are

$$\overline{T}_1 = \frac{\overline{t}(\tau) - t_n}{t_c - t_n} = 1 - \sum_{n=1}^{\infty} D_n \exp\,(-\,\mu_n^2\,\mathrm{Fo}), \tag{1.129}$$

$$\overline{\Theta}_1 = \frac{\theta_n - \overline{\theta}(\tau)}{\theta_n - \theta_p} = 1 - \sum_{n=1}^{\infty} D_n^* \exp\,(-\,\mu_n^2\,\mathrm{Fo}), \tag{1.130}$$

where D_n and D_n^* remain in the form of equation (1.82). However, C_{nj} and C_{nj}^* are determined by equation (1.126).

The average and local rates of heating and drying are easily found by differentiation of the corresponding equations with respect to the dimensionless time, Fo.

2. NON-STATIONARY FIELDS OF HEAT AND MASS TRANSFER POTENTIALS. TEMPERATURE OF THE EXTERNAL MEDIUM – A LINEAR FUNCTION OF TIME.

Real processes in most cases proceed within surroundings whose temperature is changing. A body in contact with a heat carrier heats up while the heat carrier, giving up its heat, cools down. The boundary conditions for a mathematical model of internal heat and mass transfer should reflect the temperature change in the external medium. The solutions of the system of differential equations (I) to (VI) with varying temperature of the external medium are of interest in the calculation of drying processes, coking, humidification and gasification. The importance of these solutions in the determination of heat and mass transfer coefficients is self-evident. The problems considered in this section involve an external medium whose temperature is a linear function of time. Problems where the temperature of the medium is an exponential function of time will be described in the next section.

a) Infinite Plate. Uniform Initial Conditions [$s = 0$, $\xi = x$, $t(x, 0) = t_0 =$ const, $\theta(x, 0) = \theta_0 =$ const, $t_c = t_c^0 - b\tau$]. The general solution of equations (I) to (VI), in virtue of the condition of symmetry (V), can be written in the following form:

$$t_L(x, s) - \frac{t_0}{s} =$$

$$= B_1 \operatorname{ch} \sqrt{\frac{s}{a_q}}\, \nu_1 x + B_2 \operatorname{ch} \sqrt{\frac{s}{a_q}}\, \nu_2 x, \tag{2.1}$$

$$\theta_L(x, s) - \frac{\theta_0}{s} = \frac{c}{\varepsilon\rho}(1 - \nu_1^2) B_1 \operatorname{ch} \sqrt{\frac{s}{a_q}}\, \nu_1 x +$$

$$+ \frac{c}{\varepsilon\rho}(1 - \nu_2^2) B_2 \operatorname{ch} \sqrt{\frac{s}{a_q}}\, \nu_2 x, \tag{2.2}$$

where

$$\nu_j^2 = \frac{1}{2}\left[\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right) + \right.$$

$$\left. + (-1)^j \sqrt{\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}}\,\right], \quad (j = 1, 2). \tag{2.3}$$

B_j are constants with respect to the coordinates. Their values are determined from the boundary conditions (III) to (IV). The boundary conditions, following Laplace transformations, are

$$-\lambda t_L'(R, s) + \alpha\left[\frac{t_c^0}{s} - \frac{b}{s^2} - t_L(R, s)\right] -$$

$$-(1 - \varepsilon)\rho\beta\left[\theta_L(R, s) - \frac{\theta_p}{s}\right] = 0, \tag{2.4}$$

$$\lambda_m \theta_L'(R, s) + \lambda_m \delta t_L'(R, s) +$$

$$+ \beta\left[\theta_L(R, s) - \frac{\theta_p}{s}\right] = 0. \tag{2.5}$$

Substituting $t_L(R, s)$, $t_L'(R, s)$, $\theta_L(R, s)$ and $\theta_L'(R, s)$, into equations (2.4) and (2.5), we solve the resulting algebraic system for B_1 and B_2:

$$B_1 = -\left\{\frac{(t_c^0 - t_0) P_2 + \frac{\varepsilon\rho}{c}(\theta_0 - \theta_p)(Q_2 - K_1 P_2)}{s(Q_2 P_1 - Q_1 P_2)} - \right.$$

$$
\left. - \frac{b P_2}{s^2(Q_2 P_1 - Q_1 P_2)} \right\},
$$

$$
B_2 = \frac{(t_c^0 - t_0) P_1 + \frac{\varepsilon \rho}{c} (\theta_0 - \theta_p)(Q_1 - K_1 P_1)}{s (Q_2 P_1 - Q_1 P_2)} -
$$

$$
- \frac{b P_1}{s^2(Q_2 P_1 - Q_1 P_2)},
$$

where Q_j and P_j are determined from equations (1.13) and (1.14). Consequently, the solutions of equations (2.1) and (2.2) can be rewritten in the following form:

$$
t_L(x, s) - \frac{t_0}{s} = (t_c^0 - t_0) \times
$$

$$
\times \left\{ \frac{P_1 + \varepsilon \mathrm{Ko}(Q_1 - K_1 P_1)}{s (Q_2 P_1 - Q_1 P_2)} \operatorname{ch} \sqrt{\frac{s}{a_q}} \nu_2 x - \right.
$$

$$
\left. - \frac{P_2 + \varepsilon \mathrm{Ko}(Q_2 - K_1 P_2)}{s (Q_2 P_1 - Q_1 P_2)} \operatorname{ch} \sqrt{\frac{s}{a_q}} \nu_1 x \right\} +
$$

$$
+ b \frac{P_2 \, ch \sqrt{\frac{s}{a_q}} \nu_1 x - P_1 \, ch \sqrt{\frac{s}{a_q}} \nu_2 x}{s^2(Q_2 P_1 - Q_1 P_2)} = \Pi_1 + \Pi_2, \tag{2.6}
$$

$$
\frac{\theta_0}{s} - \theta_L(x, s) = (\theta_0 - \theta_p) \times
$$

$$
\times \left\{ \frac{Q_2 + \frac{1}{\varepsilon \mathrm{Ko}} P_2 - K_1 P_2}{s (Q_2 P_1 - Q_1 P_2)} (1 - \nu_1^2) \operatorname{ch} \sqrt{\frac{s}{a_q}} \nu_1 x - \right.
$$

$$
\left. - \frac{Q_1 + \frac{1}{\varepsilon \mathrm{Ko}} P_1 - K_1 P_1}{s (Q_2 P_1 - Q_1 P_2)} (1 - \nu_2^2) \operatorname{ch} \sqrt{\frac{s}{a_q}} \nu_2 x \right\} +
$$

$$
+ \frac{c}{\varepsilon \rho} b \frac{P_1 (1 - \nu_2^2) \operatorname{ch} \sqrt{\frac{s}{a_q}} \nu_2 x - P_2 (1 - \nu_1^2) \operatorname{ch} \sqrt{\frac{s}{a_q}} \nu_1 x}{s^2 (Q_2 P_1 - Q_1 P_2)} =
$$

$$
= N_1 + N_2. \tag{2.7}
$$

The inversions of the first terms $L^{-1}[\Pi_1]$ and $L^{-1}[N_1]$ have been found in Section 1a.

We shall now determine the inversions of the remaining expressions for Π_2 and N_2. The latter describe the effect of linear temperature changes of the external medium upon the general solution. To begin with, the expression for the heat transfer potential assumes the following form:

$$\Pi_2 = b\,\frac{P_2 \operatorname{ch} \sqrt{\frac{s}{a_q}}\,\nu_1 x - P_1 \operatorname{ch} \sqrt{\frac{s}{a_q}}\,\nu_2 x}{s^2 (Q_2 P_1 - Q_1 P_2)} =$$

$$= \frac{b\,\Phi_{01}(s)}{s^2 \varphi_0(s)}. \tag{2.8}$$

The inversion of the transform will be found by means of the expansion theorem. We shall determine the roots of the denominator of equation (2.8), $s^2\varphi_0(s)$. Here we have

a) a double-zero root,

b) the roots $s = s_n$, satisfying the equation $\varphi_0(s) = Q_2P_1 - Q_1P_2 = 0$.

We can now write the expansion theorem for equation (2.8):

$$L^{-1}\left[\frac{b\,\Phi_{01}(s)}{s^2\varphi_0(s)}\right] = b \lim_{s\to 0}\left[\frac{\tau e^{s\tau}\,\Phi_{01}(s)}{\varphi_0(s)} + \right.$$

$$\left. + \frac{e^{s\tau}\,\Phi'_{01}(s)}{\varphi_0(s)} - \frac{e^{s\tau}\,\Phi_{01}(s)\,\varphi'_0(s)}{[\varphi_0(s)]^2}\right] +$$

$$+ \sum_{n=1}^{\infty} b\,\frac{\Phi_{01}(s_n)}{[s_n^2\,\varphi_0(s_n)]'}\,e^{s_n\tau}. \tag{2.9}$$

Since

$$\varphi_0(0) = \nu_2^2 - \nu_1^2,$$

$$\varphi'_0(0) = \frac{R^2}{2a_q}(\nu_2^2 - \nu_1^2) \times$$

$$\times \left[1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}} + \frac{2}{\mathrm{Bi}_q} + \frac{2}{\mathrm{Bi}_m\,\mathrm{Lu}}\right],$$

$$\Phi_{01}(0) = -(\nu_2^2 - \nu_1^2),$$

$$\Phi'_{01}(0) = -(\nu_2^2 - \nu_1^2)\frac{R^2}{2a_q} \times$$

$$\times \left(\frac{1}{\mathrm{Lu}} + \mathrm{Fe} + \frac{2}{\mathrm{Bi}_m\,\mathrm{Lu}} + \frac{x^2}{R^2}\right),$$

$$[s_n^2 \varphi_0 (s_n)]' = -\frac{a_q}{2R^2}\mu_n^3 \psi_n,$$

$$\Phi_{01}(s_n) = P_{n2}\cos \nu_1 \mu_n \frac{x}{R} - P_{n1}\cos \nu_2 \mu_n \frac{x}{R},$$

the inverted transform (2.9) can now be expressed as follows:

$$L^{-1}[\Pi_2] = -b\tau + b\frac{R^2}{2a_q}\left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right) - \\ -2\sum_{n=1}^{\infty} b\frac{R^2}{a_q\mu_n^2}\,\frac{P_{n2}\cos\nu_1\mu_n\frac{x}{R} - P_{n1}\cos\nu_2\mu_n\frac{x}{R}}{\mu_n\psi_n}\times \\ \times\exp(-\mu_n^2\mathrm{Fo}) = (t_c^0 - t_0)\left\{-\mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{2}\left(1 - \frac{x^2}{R^2} + \right.\right.\right. \\ \left.\left.+\frac{2}{\mathrm{Bi}_q}\right)\right] - 2\sum_{n=1}^{\infty}\frac{\mathrm{Pd}}{\mu_n^2}\,\frac{P_{n2}\cos\nu_1\mu_n\frac{x}{R} - P_{n1}\cos\nu_2\mu_n\frac{x}{R}}{\mu_n\psi_n}\times \\ \left.\times\exp(-\mu_n^2\mathrm{Fo})\right\}. \tag{2.10}$$

We have introduced a new dimensionless number in equation (2.10),

$$\mathrm{Pd} = \frac{bR^2}{a_q(t_c^0 - t_0)}. \tag{2.11}$$

This dimensionless number describes the rate of temperature change of the medium. A more detailed analysis shows that this dimensionless number is a special case of the expression $\left|\frac{dT_c}{d\mathrm{Fo}}\right|_{max}$. The latter is called the Predvoditelev number. Thus, equation (2.11) is an expression for the Predvoditelev number for the case of linear time dependence of the temperature of the medium. In this dimensionless number the constant b denotes the rate of change of the temperature of the external medium, and $t_c^0=\mathrm{const}$ denotes the temperature of the medium at time zero.

We shall now find the inverse of the transform for N_2:

$$N_2 = \frac{c}{\varepsilon\rho}\, b\, \frac{P_1(1-\nu_2^2)\operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x - P_2(1-\nu_1^2)\operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x}{s^2(Q_2P_1 - Q_1P_2)} =$$

$$= \frac{c}{\varepsilon\rho}\, b\, \frac{\Phi_{02}(s)}{s^2\varphi_0(s)}. \tag{2.12}$$

The roots of the denominator of equation (2.12) remain as before. Therefore,

$$L^{-1}\left[\frac{c}{\varepsilon\rho}\, b\, \frac{\Phi_{02}(s)}{s^2\varphi_0(s)}\right] = \frac{c}{\varepsilon\rho}\, b \lim_{s\to 0}\left[\tau e^{s\tau}\frac{\Phi_{02}(s)}{\varphi_0(s)} + \right.$$

$$\left. + e^{s\tau}\frac{\Phi_{02}'(s)}{\varphi_0(s)} - e^{s\tau}\frac{\Phi_{02}(s)\,\varphi_0'(s)}{[\varphi_0(s)]^2}\right] + \frac{c}{\varepsilon\rho}\, b \times$$

$$\times \sum_{n=1}^{\infty}\frac{\Phi_{02}(s_n)}{[s_n^2\varphi_0(s_n)]'}\exp(s_n\tau) = \frac{c}{\varepsilon\rho}\, b\,\mathrm{Fe}\,\frac{R^2}{2a_q}\times$$

$$\times\left(1-\frac{x^2}{R^2}\right) - \sum_{n=1}^{\infty} 2\,\frac{c}{\varepsilon\rho}\,\frac{bR^2}{a_q\mu_n^2}\times$$

$$\times\frac{P_{n1}(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R} - P_{n2}(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R}}{\mu_n\psi_n}\times$$

$$\times\exp(-\mu_n^2\mathrm{Fo}) = (\theta_0-\theta_p)\left\{\frac{1}{2}\mathrm{PnPd}\left(1-\frac{x^2}{R^2}\right) - \frac{2}{\varepsilon\mathrm{Ko}}\times\right.$$

$$\times\sum_{n=1}^{\infty}\frac{\mathrm{Pd}}{\mu_n^2}\,\frac{P_{n1}(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R} - P_{n2}(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R}}{\mu_n\psi_n}\times$$

$$\left.\times\exp(-\mu_n^2\mathrm{Fo})\right\}, \tag{2.13}$$

because

$$\Phi_{02}(0) = 0$$

$$\Phi_{02}'(0) = -(\nu_2^2-\nu_1^2)\,\mathrm{Fe}\,\frac{R^2}{2a_q}\left(\frac{x^2}{R^2}-1\right),$$

$$\Phi_{02}(s_n) = P_{n1}(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R} - P_{n2}(1-\nu_1^2)\times$$

$$\times\cos\nu_1\mu_n\frac{x}{R}.$$

We can now present the final solution of our problem:

$$T=\frac{t(x,\tau)-t_0}{t_c^0-t_0}=1-\mathrm{Pd}\left[\mathrm{Fo}-\frac{1}{2}\left(1-\frac{x^2}{R^2}+\frac{2}{\mathrm{Bi}_q}\right)\right]-$$

$$-\sum_{n=1}^{\infty}\left(C_{n2}\cos\nu_1\mu_n\frac{x}{R}-C_{n1}\cos\nu_2\mu_n\frac{x}{R}\right)\times$$

$$\times\exp(-\mu_n^2\,\mathrm{Fo}); \tag{2.14}$$

$$\Theta=\frac{\theta_0-\theta(x,\tau)}{\theta_0-\theta_p}=1+\frac{1}{2}\mathrm{PnPd}\left(1-\frac{x^2}{R^2}\right)-$$

$$-\sum_{n=1}^{\infty}\left[C_{n1}^{*}\,(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R}-C_{n2}^{*}\times\right.$$

$$\left.\times(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R}\right]\exp(-\mu_n^2\,\mathrm{Fo}), \tag{2.15}$$

where

$$C_{nj}=2\frac{\left(1-\varepsilon\mathrm{Ko}K_1+\frac{\mathrm{Pd}}{\mu_n^2}\right)P_{nj}+\varepsilon\mathrm{Ko}Q_{nj}}{\mu_n\psi_n},$$

$$C_{nj}^{*}=\frac{1}{\varepsilon\mathrm{Ko}}C_{nj}. \tag{2.16}$$

Q_{nj}, P_{nj}, ψ_n are determined by equations (1.21) and (1.23), and the roots are determined by the characteristic equation (1.20). The dimensionless numbers remain the same as in the problem for $t_c=\text{const}$, with the exception that t_c is substituted for t_c^0.

The average dimensionless temperature and mass transfer potentials are obtained in the usual manner:

$$\bar{T}=\frac{\bar{t}(\tau)-t_0}{t_c^0-t_0}=1-\mathrm{Pd}\left[\mathrm{Fo}-\left(\frac{1}{3}+\frac{1}{\mathrm{Bi}_q}\right)\right]-$$

$$-\sum_{n=1}^{\infty}D_n\exp(-\mu_n^2\,\mathrm{Fo}), \tag{2.17}$$

$$\bar{\Theta} = \frac{\theta_0 - \theta(\tau)}{\theta_0 - \theta_p} = 1 + \frac{1}{3}\mathrm{PnPd} - \sum_{n=1}^{\infty} D_n^* \exp(-\mu_n^2 \mathrm{Fo}), \tag{2.18}$$

where

$$D_n = D_{n2} - D_{n1}, \quad D_n^* = D_{n1}^* - D_{n2}^*.$$

The expressions for D_{nj} and D_{nj}^* remain the same as in the previous problems where average values for an infinite plate were determined. In this case, however, equation (2.16) has to be taken into account.

The local and average dimensionless rates of change of the transfer potentials are

$$\frac{dT}{d\mathrm{Fo}} = \sum_{n=1}^{\infty}\left(C_{n2}\cos\nu_1\mu_n\frac{x}{R} - C_{n1}\cos\nu_2\mu_n\frac{x}{R}\right)\times \mu_n^2\exp(-\mu_n^2\mathrm{Fo}) - \mathrm{Pd}, \tag{2.19}$$

$$\frac{d\bar{T}}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n\mu_n^2\exp(-\mu_n^2\mathrm{Fo}) - \mathrm{Pd}, \tag{2.20}$$

$$\frac{d\Theta}{d\mathrm{Fo}} = \sum_{n=1}^{\infty}\left[C_{n1}^*(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R} - C_{n2}^*\times(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R}\right]\mu_n^2\exp(-\mu_n^2\mathrm{Fo}), \tag{2.21}$$

$$\frac{d\bar{\Theta}}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n^*\mu_n^2\exp(-\mu_n^2\mathrm{Fo}). \tag{2.22}$$

Analysis of the solutions of equations (2.14), (2.15) and (2.17) to (2.22) shows that the starting Fo values (when only the first two terms of the infinite series are used) remain the same as in a constant temperature medium up to Pd $\approx$ 0.25 (see Table 18). An increase of Pd results in a slow decrease in the Fo values. With increasing dimensionless time, there is a diminishing influence of the infinite series upon the value of the dimensionless potentials.

Table 18. Starting values of Fo for simplified calculations of heat and mass transfer in an infinite plate with $t_c = t_c^0 - b\tau$

Index		Fo	
		$/\Delta/$ %	
		1,0 %	0,5 %
$1-T$	$x/R=0$	0,73	0,88
	$x/R=1$	0,93	1,09
$1-\Theta$	$x/R=0$	0,79	0,95
	$x/R=1$	1,04	
$1-\overline{\Theta}$		0,12	0,17
$1-\overline{\Theta}$		0,14	0,30
$d\overline{T}/d\mathrm{Fo}$		0,26	0,35
$\overline{d\Theta}/d\mathrm{Fo}$		1,04	

For $T\,(\mathrm{Pd} = 0{,}25,\ \mathrm{Bi}_q = 10,\ \frac{x}{R} = 0,\ \varepsilon = 0{,}5,\ \mathrm{Pn} = 0{,}5,\ \mathrm{Ko} = 1{,}2,$ $\mathrm{Bi}_m = 10)$, the magnitude of the transient term in the solution for T as a percentage of the remaining terms is as follows:

for Fo = 5.0, 23.7%
for Fo = 7.0, 1.31%
for Fo = 10.0, 0.110%

Beyond a value of Fo = $\mathrm{Fo}_{\text{quasistationary}}$, the dimensionless potentials are determined only by the steady-state terms. Then the temperature at any point of the plate becomes a linear function of time, and the temperature distribution through the thickness of the material becomes parabolic. Consequently, starting from $\mathrm{Fo}_{\text{quasistationary}}$ we shall have similar temperature distributions. This regime is called quasistationary. At the same time the redistribution of the mass potential ceases and a stationary parabolic distribution of the mass potential is established. As Pd increases, the quasistationary regime takes place at lower values of Fo.

Thus, for $\mathrm{Fo} \geqslant \mathrm{Fo}_{\text{quasistationary}}$, equations (2.14), (2.15), (2.17) and (2.18) are:

$$T = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{2}\left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right], \tag{2.14'}$$

$$\Theta = 1 + \frac{1}{2}\ \mathrm{Pn}\,\mathrm{Pd}\left(1 - \frac{x^2}{R^2}\right), \tag{2.15'}$$

$$\overline{T} = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \left(\frac{1}{3} + \frac{1}{\mathrm{Bi}_q}\right)\right], \tag{2.17'}$$

$$\overline{\Theta} = 1 + \frac{1}{3}\,\mathrm{PnPd}. \tag{2.18'}$$

To simplify calculations for a medium with a linearly decreasing temperature, the heat and mass transfer coefficients, C_{nj}, C^{*}_{nj}, D_n and D^{*}_{n}, are given as functions of the Predvoditelev number in Figures 23 and 24. Tables 19 to 22 contain the same data in numerical form.

b) Sphere. Uniform Initial Conditions.

$$[\Gamma=2,\ \xi=r,\ t(r,0)=t_0=\mathrm{const},\ \theta(r,0)=\theta_0=\mathrm{const},$$
$$t_c=t_c^0-b\tau].$$

The general transform solution of the problem can be written as follows:

$$\vartheta_L(r,s)-\frac{rt_0}{s}=B_1\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r+$$
$$+B_2\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r,$$

$$u_L(r,s)-\frac{r\theta_0}{s}=B_1\frac{c}{\varepsilon\rho}(1-\nu_1^2)\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r+$$
$$+B_2\frac{c}{\varepsilon\rho}(1-\nu_2^2)\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r. \tag{2.23}$$

The constants, B_j, are determined from the boundary conditions (III) to (IV), which can be expressed by equations (2.4) and (2.5). We satisfy the boundary conditions in equations (2.22) and (2.23) and solve the resulting system of equations for B_1 and B_2. The results can now be substituted into equations (2.22) and (2.23):

$$\vartheta_L(r,s)-\frac{rt_0}{s}=(t_c^0-t_0)\times$$

$$\times\left\{\frac{\widetilde{P}_1+\varepsilon\mathrm{Ko}(\widetilde{Q}_1-K_1\widetilde{P}_1)}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r-\right.$$
$$\left.-\frac{\widetilde{P}_2+\varepsilon\mathrm{Ko}(\widetilde{Q}_2-K_1\widetilde{P}_2)}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r\right\}+$$

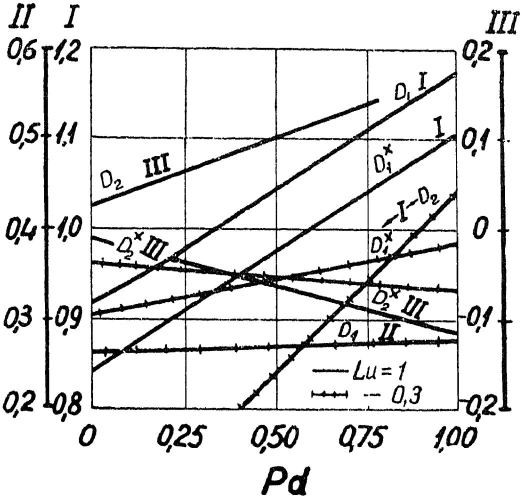

Fig. 23. Heat and mass transfer coefficients as a function of Pd (infinite plate, $t_c = t_c^0 - b\tau$).

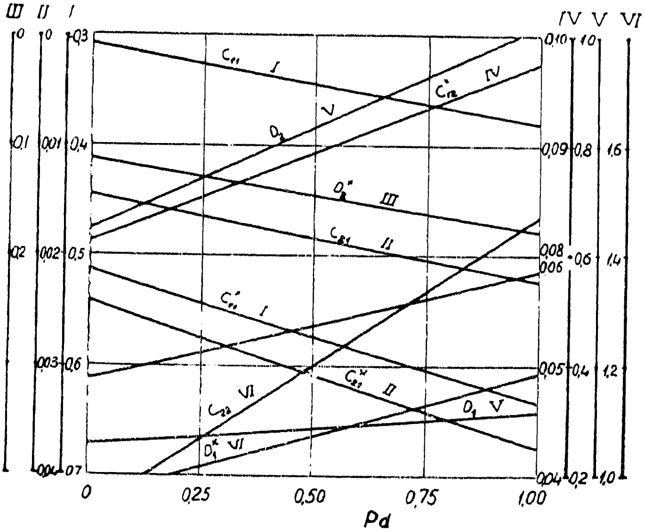

Fig. 24. Heat and mass transfer coefficients as a function of Pd (infinite plate, $t_c = t_c^0 - b\tau$).

Table 19. Heat and mass transfer coefficients as a function of $\mathrm{Pd}=\frac{bR^2}{a_q\left(t_c^0-t_0\right)}$ (infinite plate*)

Pd	C_{n1}		C_{n2}		C^*_{n1}		C^*_{n2}		D_n		D^*_n	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—0,3080	—0,01452	0,04880	0,9378	—0,5134	—0,02420	0,08130	1,564	0,2570	0,6500	0,9616	—0,1137
0,1	—0,3152	—0,01532	0,04976	0,9908	—0,5254	—0,02554	0,08295	1,652	0,2628	0,6867	0,9842	—0,1202
0,2	—0,3225	—0,01612	0,05073	1,044	—0,5376	—0,02687	0,08457	1,740	0,2687	0,7235	1,007	—0,1265
0,25	—0,3261	—0,01652	0,05125	1,070	—0,5436	—0,02754	0,08513	1,784	0,2716	0,7416	1,018	—0,1297
0,3	—0,3297	—0,01692	0,05169	1,097	—0,5496	—0,02821	0,08617	1,829	0,2745	0,7603	1,029	—0,1330
0,4	—0,3369	—0,01772	0,05266	1,150	—0,5616	—0,02954	0,08778	1,917	0,2804	0,7971	1,052	—0,1394
0,5	—0,3441	—0,01852	0,05362	1,203	—0,5736	—0,03087	0,08938	2,005	0,2862	0,8338	1,075	—0,1458
0,6	—0,3514	—0,01932	0,05458	1,256	—0,5858	—0,03221	0,09098	2,094	0,2921	0,8706	1,098	—0,1523
0,7	—0,3586	—0,02012	0,05555	1,309	—0,5978	—0,03354	0,09260	2,182	0,2980	0,9072	1,120	—0,1587
0,75	—0,3622	—0,02052	0,05603	1,355	—0,6038	—0,03421	0,09340	2,225	0,3009	0,9252	1,132	—0,1618
0,8	—0,3658	—0,02092	0,05651	1,362	—0,6098	—0,03487	0,09420	2,270	0,3039	0,9440	1,143	—0,1650
0,9	—0,3730	—0,02172	0,05748	1,414	—0,6218	—0,03621	0,09582	2,357	0,3097	0,9800	1,165	—0,1714
1,0	—0,3803	—0,02252	0,05844	1,467	—0,6340	—0,03754	0,09742	2,445	0,3156	1,031	1,88	—0,1778

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*Calculations performed for $\mathrm{Lu}=0{,}3$; $\varepsilon=0{,}5$; $\mathrm{Ko}=1{,}2$; $\mathrm{Pn}=0{,}5$; $\mathrm{Bi}_m=10$; $\mathrm{Bi}_q=10$. and uniform initial conditions.

Table 20. Heat and mass transfer coefficients as a function of $\mathrm{Pd} = \frac{bR^2}{a_q(t_c^0 - t_0)}$ (infinite plate) $\mathrm{Lu} = 1{,}0$, $\varepsilon = 0{,}5$, $\mathrm{Ko} = 1{,}2$, $\mathrm{Pn} = 0{,}5$, $\mathrm{Bi}_m = \mathrm{Bi}_q = 10$

dd	C_{n1}		C_{n2}		C_{n1}^*		C_{n2}^*		D_n		D_n^*	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—1,135	—0,003402	0,1399	0,03438	—1,892	—0,00572	0,2334	0,0572	0,9009	0,02439	0,8436	—0,01538
0,25	—1,222	—0,007787	0,1498	0,08778	—2,037	—0,01298	0,2497	0,1463	0,9692	0,06206	0,909	—0,03962
0,50	—1,309	—0,01217	0,1596	0,1412	—2,182	—0,02029	0,2661	0,2354	1,037	0,09975	0,974	—0,06386
0,75	—1,396	—0,01656	0,1695	0,1946	—2,327	—0,02761	0,2826	0,3244	1.106	0,1374	1,039	—0,08806
1,00	—1,483	—0,02094	0,1793	0,2480	—2,472	—0,03491	0,2989	0,4134	1,174	0,1751	1,104	—0,1123

Table 21. Heat and mass transfer coefficients as a function of Pd (infinite plate) $\mathrm{Lu}=0{,}3$, $\varepsilon=0{,}5$, $\mathrm{Ko}=1{,}2$, $\mathrm{Pn}=0{,}1667$, $\mathrm{Bi}_m=10$, $\mathrm{Bi}_q=10$

PP	C_{n1}		C_{n2}		C_{n1}^*		C_{n2}		D_n		D_n^*	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
0	—0,3188	—0,005276	0,04618	0,9158	—0,5312	—0,008794	0,07694	1,527	0,2620	0,6353	0,9045	—0,03952
0,25	—0,3258	—0,006395	0,04536	1,062	—0,5431	—0,01066	0,07562	1,770	0,2662	0,7368	0,9248	—0,04563
0,50	—0,3327	—0,006914	0,04455	1,207	—0,5546	—0.01153	0,07426	2,012	0,2702	0,8372	0,9446	—0,05209
0,75	—0,3397	—0,007733	0,04373	1,353	—0,5663	—0,01289	0,07290	2,255	0,2743	0,9386	0,9645	—0,05839
1,00	—0,3466	—0,008552	0,04292	1,499	—0,5778	—0,01426	0,07155	2,499	0,2783	1,040	0,9843	—0,06471

Table 22. Heat and mass transfer coefficients as a function of the Biot number (Bi_q) (infinite plate) $Lu=0,3$, $\varepsilon=0,5$, $Ko=1,2$, $Pn=0,5$, $Bi_m=10$, $Pd=0,25$

Bi_q	C_{n1}		C_{n2}		C^*_{n1}		C^*_{n2}		D_n		D^*_n	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
5	—0,3295	—0,00990	0,1643	1,010	—0,5493	—0,01650	0,2739	1,684	0,3789	0,7390	1,013	—0,1278
7	—0,3274	—0,01266	0,09726	1,054	—0,5458	—0,02110	0,1621	1,757	0,3152	0,7500	1,016	—0,1303
10	—0,3261	—0,01652	0,05125	1,070	—0,5436	—0,02754	0,08543	1,784	0,2716	0,7416	1,018	—0,1297
15	—0,3251	—0,01773	0,01597	1,079	—0,5419	—0,02956	0,02662	1,799	0,2383	0,7332	1,019	—0,1299
20	—0,3243	—0,01860	—0,003319	1,088	—0,5406	—0,03101	0,005533	1,814	0,2197	0,7314	1,019	—0,1305

$$+b\,\frac{\widetilde{P}_2\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r-\widetilde{P}_1\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r}{s^2(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}=\Pi_1+\Pi_2, \tag{2.24}$$

$$\frac{r\theta_0}{s}-u_L(r,s)=(\theta_0-\theta_p)\times$$

$$\times\left\{\frac{\widetilde{Q}_2+\frac{1}{\varepsilon\mathrm{Ko}}\widetilde{P}_2-K_1\widetilde{P}_2}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}(1-\nu_1^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r-\right.$$

$$\left.-\frac{\widetilde{Q}_1+\frac{1}{\varepsilon\mathrm{Ko}}\widetilde{P}_1-K_1\widetilde{P}_1}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}(1-\nu_2^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r\right\}-$$

$$-b\,\frac{c}{\varepsilon\rho}\times$$

$$\times\frac{\widetilde{P}_2(1-\nu_1^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r-\widetilde{P}_1(1-\nu_2^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r}{s^2(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}=$$

$$=N_1+N_2. \tag{2.25}$$

$\widetilde{Q}_j$ and $\widetilde{P}_j$ are determined from equations (1.60) and (1.61); ν_j is determined by equation (2.3). The inversions of Π_1 and N_1 have been found in Section 1b.

We can now determine the inversions for Π_2 and N_2:

$$\Pi_2=b\,\frac{\widetilde{P}_2\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r-\widetilde{P}_1\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r}{s^2(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}=$$

$$=b\,\frac{\Phi_1(s)}{\psi_1(s)}. \tag{2.26}$$

The numerator and denominator of equation (2.26) will be put into a generalized form by dividing both parts of equation (2.26) by

$$\frac{s}{a_q}:\Phi_1(s):\frac{s}{a_q}=\Phi_{01}(s),\ \psi_1(s):\frac{s}{a_q}=\psi_0(s).$$

Since the roots of the denominator are a double zero root and $s_n=-\frac{a_q\mu_n^2}{R^2}$, corresponding to the equation $\frac{\psi_0(s)}{s^2}=\varphi_0(s)=0$, the generalized expansion theorem can be expressed as

$$L^{-1}[\Pi_2] = L^{-1}\left[b\,\frac{\Phi_1(s)}{\psi_1(s)} \right] = b\,\lim_{s\to 0}\left[\tau\, e^{s\tau}\,\frac{\Phi_{01}(s)}{\varphi_0(s)} + \right.$$

$$\left. + e^{s\tau}\,\frac{\Phi'_{01}(s)}{\varphi_0(s)} - e^{s\tau}\,\frac{\Phi_{01}(s)\varphi'_0(s)}{[\varphi_0(s)]^2} \right] +$$

$$+ \sum_{n=1}^{\infty} b\,\frac{\Phi_1(s_n)}{\psi'_1(s_n)}\exp(s_n\tau) =$$

$$= -b\tau r + b\,\frac{R^2}{6a_q}\,r\left(1 - \frac{r^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right) +$$

$$+ \sum_{n=1}^{\infty} 2\,\frac{b}{a_q}\,\frac{R^2}{\mu_n^2}\left(\frac{P_{n2} R \sin \nu_1 \mu_n \frac{r}{R}}{\mu_n \psi_n} - \right.$$

$$\left. - \frac{P_{n1} R \sin \nu_2 \mu_n \frac{r}{R}}{\mu_n \psi_n} \right) \exp(-\mu_n^2\,\mathrm{Fo}) =$$

$$= r(t_c^0 - t_0)\left\{ -\mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{6}\left(1 - \frac{r^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right] - \right.$$

$$-2\sum_{n=1}^{\infty} \frac{\mathrm{Pd}}{\mu_n^2}\,\frac{P_{n1} R \sin \nu_2 \mu_n \frac{r}{R} - P_{n2} R \sin \nu_1 \mu_n \frac{r}{R}}{\mu_n \psi_n} \times$$

$$\left. \times \exp(-\mu_n^2\,\mathrm{Fo}) \right\}, \qquad (2.27)$$

because $\varphi_0(0) = (\nu_2^2 - \nu_1^2)\,\nu_1\nu_2$,

$$\varphi'_0(0) = \frac{R^2}{6a_q}\,\nu_1\nu_2\left(1 + \frac{1}{\mathrm{Lu}} + \mathrm{Fe} + \right.$$

$$\left. + \frac{2}{\mathrm{Bi}_q} + \frac{2}{\mathrm{Bi}_m \mathrm{Lu}}\right)(\nu_2^2 - \nu_1^2),$$

$$\Phi_{01}(0) = -r\,(\nu_2^2 - \nu_1^2)\,\nu_1\nu_2,$$

$$\Phi'_{01}(0) = r\,\frac{R^2}{6a_q}\,\nu_1\nu_2 \times$$

$$\times\left(-\frac{2}{\mathrm{Bi}_m \mathrm{Lu}} - \frac{1}{\mathrm{Lu}} - \mathrm{Fe} - \frac{r^2}{R^2}\right)(\nu_2^2 - \nu_1^2),$$

$$\psi_1'(s_n) = -\frac{a_q \mu_n^3}{2R^4}\psi_n ,$$

$$\Phi_1(s_n) = -\frac{1}{R}\left(P_{n2}\sin\nu_1\mu_n\frac{r}{R} - P_{n1}\sin\nu_2\mu_n\frac{r}{R}\right),$$

the inversion

$$N_2 = -b\frac{c}{\varepsilon\rho}\times$$

$$\times\frac{\widetilde{P}_2(1-\nu_1^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_1 r - \widetilde{P}_1(1-\nu_2^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_2 r}{s^2(\widetilde{Q}_2\widetilde{P}_1 - \widetilde{Q}_1\widetilde{P}_2)} =$$

$$= -\frac{bc}{\varepsilon\rho}\frac{\Phi_2(s)}{\psi_1(s)}$$

can be obtained by the same method as was used for the inversion of Π_2. Calculations yield

$$L^{-1}[N_2] = \frac{c}{\varepsilon\rho}b\frac{R^2}{6a_q}r\,\mathrm{Fe}\left(1-\frac{r^2}{R^2}\right) -$$

$$-2\sum_{n=1}^{\infty}\frac{c}{\varepsilon\rho}b\frac{R^2}{a_q\mu_n^2}\times$$

$$\times\frac{P_{n2}(1-\nu_1^2)R\sin\nu_1\mu_n\frac{r}{R} - P_{n1}(1-\nu_2^2)R\sin\nu_2\mu_n\frac{r}{R}}{\mu_n\psi_n}e^{-\mu_n^2\mathrm{Fo}} =$$

$$= (\theta_0-\theta_p)r\left\{\frac{1}{6}\mathrm{Pn}\,\mathrm{Pd}\left(1-\frac{r^2}{R^2}\right) -\right.$$

$$-2\sum_{n=1}^{\infty}\frac{\mathrm{Pd}}{\varepsilon\,\mathrm{Ko}\mu_n^2}\times$$

$$\times\left[\frac{P_{n2}(1-\nu_1^2)\sin\nu_1\mu_n^2\frac{r}{R} - P_{n1}(1-\nu_2^2)\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}\mu_n\psi_n}\right]\times$$

$$\times\exp(-\mu_1^2\mathrm{Fo}).$$

Thus, the solution of the system of differential equations for internal heat and mass transfer, and for heat and mass transfer potentials, can be expressed in the following form, by use of the inversions (2.27) and (2.28) and the solutions obtained in Section 1b:

$$T=\frac{t(r,\tau)-t_0}{t_c^0-t_0}=1-\mathrm{Pd}\left[\mathrm{Fo}-\frac{1}{6}\left(1-\frac{r^2}{R^2}+\frac{2}{\mathrm{Bi}_q}\right)\right]-$$

$$-\sum_{n=1}^{\infty}\left(C_{n1}\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}}-C_{n2}\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}}\right)\times$$

$$\times\exp(-\mu_n^2\,\mathrm{Fo}), \tag{2.29}$$

$$\Theta=\frac{\theta_0-\theta(r,\tau)}{\theta_0-\theta_p}=1+\frac{1}{6}\,\mathrm{Pn}\,\mathrm{Pd}\left(1-\frac{r^2}{R^2}\right)-$$

$$-\sum_{n=1}^{\infty}\left[C_{n2}^{*}(1-\nu_1^2)\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}}-\right.$$

$$\left.-C_{n1}^{*}(1-\nu_2^2)\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}}\right]\exp(-\mu_n^2\,\mathrm{Fo}). \tag{2.30}$$

Here, C_{nj} and C_{nj}^{*} are determined by equation (2.16); Q_{nj}, P_{nj} and ψ_n are determined by equations (1.66), (1.67) and (1.70).

The average dimensionless potentials for a sphere can now be expressed by equations (2.31) and (2.32):

$$\overline{T}=\frac{\overline{t}(\tau)-t_0}{t_c^0-t_0}=1-\mathrm{Pd}\left[\mathrm{Fo}-\frac{1}{15}\left(1+\frac{5}{\mathrm{Bi}_q}\right)\right]-$$

$$-\sum_{n=1}^{\infty}D_n\exp(-\mu_n^2\mathrm{Fo}), \tag{2.31}$$

$$\overline{\Theta}=\frac{\theta_0-\overline{\theta}(\tau)}{\theta_0-\theta_p}=1+\frac{1}{15}\mathrm{Pn}\,\mathrm{Pd}-$$

$$-\sum_{n=1}^{\infty}D_n^{*}\exp(-\mu_n^2\,\mathrm{Fo}), \tag{2.32}$$

where $D_n = D_{n1} - D_{n2}$, $D_n^* = D_{n2}^* - D_{n1}^*$, D_{nj} and D_{nj}^* have the usual form for a sphere, including, however, a new expression for C_{nj} and C_{nj}^*.

The characteristic roots, μ_n, in equations (2.27) to (2.32) remain the same as in the problem with t_c= constant.

The solutions for both the sphere and the plate show that the dimensionless temperature and the mass transfer potentials at any point of the body depend on the Predvoditelev number. They also depend on the rate of change of temperature of the external medium. However, as a result of the comparatively rapid convergence of the infinite series, the Predvoditelev number can be neglected starting with $\mathrm{Fo} = \mathrm{Fo}_{\text{quasistationary}}$. In this case the distribution of the potentials becomes parabolic, and the change of T with respect to time becomes linear (at Bi_q= const), i.e., for $\mathrm{Fo} \geqslant \mathrm{Fo}_{\text{quasistationary}}$.

$$T = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{6}\left(1 - \frac{r^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right], \tag{2.29'}$$

$$\Theta = 1 + \frac{1}{6}\,\mathrm{Pn}\,\mathrm{Pd}\left(1 - \frac{r^2}{R^2}\right), \tag{2.30'}$$

$$\overline{T} = 1 - \mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{15}\left(1 + \frac{5}{\mathrm{Bi}_q}\right)\right], \tag{2.31'}$$

$$\overline{\Theta} = 1 + \frac{1}{15}\,\mathrm{Pn}\,\mathrm{Pd}. \tag{2.32'}$$

c) Infinite plate. Parabolic Initial Conditions

$$\left[\Gamma = 0,\ \xi = x,\ t(x,0) = t_{ц} + \left(\frac{x}{R}\right)^2 (t_n - t_{ц}),\right.$$
$$\left.\theta(x,0) = \theta_{ц} - \left(\frac{x}{R}\right)^2 (\theta_{ц} - \theta_n),\ \ t_c = t_c^0 - b\tau\ \right].$$

The general transform solution of the system of differential equations for heat and mass transfer with parabolic initial conditions has the following form:

$$t_L(x, s) = \frac{t_{ц} - (t_{ц} - t_n)\dfrac{x^2}{R^2}}{s} + \frac{2a_m\omega}{s^2} +$$
$$+ B_1 \operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x + B_2 \operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x, \tag{2.33}$$

$$\theta_L(x,s)=\frac{\theta_{\text{ц}}-(\theta_{\text{ц}}-\theta_n)\frac{x^2}{R^2}}{s}+\frac{2\frac{a_q c}{\varepsilon\rho}\left(\frac{t_{\text{ц}}-t_n}{R^2}+\omega\,\mathrm{Lu}\right)}{s^2}+$$
$$+B_1\frac{c}{\varepsilon\rho}(1-\nu_1^2)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x+$$
$$+B_2\frac{c}{\varepsilon\rho}(1-\nu_2^2)\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x. \tag{2.34}$$

In equations (2.33) and (2.34) B_1 and B_2 are constants, ν_j is expressed in the usual way and ω is determined by equation (1.91).

The B_j, found from boundary conditions (2.4) and (2.5), have the following form:

$$B_1=-\left[\frac{\Omega_1 P_2-\Omega_3 Q_2}{s(Q_2P_1-Q_1P_2)}+\frac{\Omega_2' P_2-\Omega_4 Q_2}{s^2(Q_2P_1-Q_1P_2)}\right],$$
$$B_2=\frac{\Omega_1 P_1-\Omega_3 Q_1}{s(Q_2P_1-Q_1P_2)}+\frac{\Omega_2' P_1-\Omega_4 Q_1}{s^2(Q_2P_1-Q_1P_2)},$$

where Q_j, P_j and Ω_k have the same values as given in Section 1c:

$$\Omega_2'=-2a_q\left[K_1\left(\frac{t_{\text{ц}}-t_n}{R^2}+\omega\,\mathrm{Lu}\right)+\omega\,\mathrm{Lu}+\frac{b}{2a_q}\right].$$

Consequently, the transform solution of equations (I)-(VI) is

$$t_L(x,s)=\frac{t_{\text{ц}}-(t_{\text{ц}}-t_n)\frac{x^2}{R^2}}{s}+\frac{2a_m\omega}{s^2}+\left\{\left[\frac{\Omega_1 P_1-\Omega_3 Q_1}{s(Q_2P_1-Q_1P_2)}+\right.\right.$$
$$\left.+\frac{\Omega_2' P_1-\Omega_4 Q_1}{s^2(Q_2P_1-Q_1P_2)}\right]\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x-\left[\frac{\Omega_1 P_2-\Omega_3 Q_2}{s(Q_2P_1-Q_1P_2)}+\right.$$
$$\left.\left.+\frac{\Omega_2' P_2-\Omega_4 Q_2}{s^2(Q_2P_1-Q_1P_2)}\right]\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x\right\}, \tag{2.35}$$

$$\theta_L(x,s)=\frac{\theta_{\text{ц}}-(\theta_{\text{ц}}-\theta_n)\frac{x^2}{R^2}}{s}+\frac{2\frac{a_q c}{\varepsilon\rho}\left(\frac{t_{\text{ц}}-t_n}{R^2}+\omega\,\mathrm{Lu}\right)}{s^2}+$$
$$+\left\{\left[\frac{\Omega_1 P_1-\Omega_3 Q_1}{s(Q_2P_1-Q_1P_2)}+\frac{\Omega_2' P_1-\Omega_4 Q_1}{s^2(Q_2P_1-Q_1P_2)}\right]\frac{c}{\varepsilon\rho}(1-\nu_2^2)\times\right.$$
$$\times\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x-\left[\frac{\Omega_1 P_2-\Omega_3 Q_2}{s(Q_2P_1-Q_1P_2)}+\frac{\Omega_2' P_2-\Omega_4 Q_2}{s^2(Q_2P_1-Q_1P_2)}\right]\times$$

$$\times \frac{c}{\varepsilon\rho}\left(1-\nu_1^2\right)\operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x\Bigg\} . \tag{2.36}$$

The inversions of the basic expressions contained in equations (2.35) and (2.36), were obtained in Section 1c. Substituting their values for the inverse transformations, $L^{-1}[t_L(x, s)]$ and $L^{-1}[\theta_L(x, s)]$ and simplifying, we get

$$T_1=\frac{t(x,\tau)-t_n}{t_c^0-t_n}=1-\text{Pd}\left[\text{Fo}-\frac{1}{2}\left(1-\frac{x^2}{R^2}+\frac{2}{\text{Bi}_q}\right)\right]-$$

$$-\sum_{n=1}^{\infty}\left(C_{n2}\cos\nu_1\mu_n\frac{x}{R}-C_{n1}\cos\nu_2\mu_n\frac{x}{R}\right)\exp(-\mu_n^2\,\text{Fo}), \tag{2.37}$$

$$\Theta_1=\frac{\theta_n-\theta(x,\tau)}{\theta_n-\theta_p}=1+\frac{1}{2}\,\text{Pd}\,\text{Pn}\left(1-\frac{x^2}{R^2}\right)-$$

$$-\sum_{n=1}^{\infty}\left[C_{n1}^{*}\left(1-\nu_2^2\right)\cos\nu_2\mu_n\frac{x}{R}-C_{n2}^{*}\left(1-\nu_1^2\right)\cos\nu_1\mu_n\frac{x}{R}\right]\times$$

$$\times\exp\left(-\mu_n^2\,\text{Fo}\right), \tag{2.38}$$

where

$$C_{nj}=2\,\frac{G_1P_{nj}-G_2Q_{nj}}{\mu_n\,\psi_n}\,,\quad C_{nj}^{*}=\frac{1}{\varepsilon\text{Ko}}C_{nj}\,, \tag{2.39}$$

$$G_1=1-2W\left(\frac{1}{\text{Bi}_q}-\frac{1}{\mu_n^2}\right)-\varepsilon\,\text{Ko}\,K_1-$$

$$-2\varepsilon\,\text{Ko}K_2\frac{(K_1+1)\,\text{Lu}}{\mu_n^2}+\frac{\text{Pd}}{\mu_n^2}\,, \tag{2.40}$$

$$G_2=2\varepsilon\,\text{Ko}K_2\left(\frac{1}{\text{Bi}_m}-\frac{\text{Lu}}{\mu_n^2}\right)-\varepsilon\,\text{Ko}. \tag{2.41}$$

The expression s for the remaining quantities are given in Section 1c.

d) Sphere. Parabolic Initial Conditions

$$\Bigg[\Gamma=2,\ \xi=r,\ t(r,o)=t_{\text{ц}}+\left(\frac{r}{R}\right)^2(t_n-t_{\text{ц}}),$$

$$\theta(r,0)=\theta_{\text{ц}}-\left(\frac{r}{R}\right)^2(\theta_{\text{ц}}-\theta_n),\ t_c=t_c^0-b\tau\Bigg].$$

The transform solution for a sphere may be written as follows:

$$\vartheta_L(r,\ s)=\frac{r\left[t_{\text{ц}}-(t_{\text{ц}}-t_n)\frac{r^2}{R^2}\right]}{s}+\frac{6r\,a_m\,\omega}{s^2}+$$

$$+\left\{\left[\frac{\Omega_1\widetilde{P}_1-\Omega_2\widetilde{Q}_1}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}+\frac{\Omega_2'\widetilde{P}_1-\Omega_4\widetilde{Q}_1}{s^2(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\right]\times\right.$$

$$\times\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r-\left[\frac{\Omega_1\widetilde{P}_2-\Omega_3\widetilde{Q}_2}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}+\right.$$

$$\left.\left.+\frac{\Omega_2'\widetilde{P}_2-\Omega_4\widetilde{Q}_2}{s^2(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\right]\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r\right\}, \tag{2.42}$$

$$u_L(r,s)=\frac{r\left[\theta_{\text{ц}}-(\theta_{\text{ц}}-\theta_n)\frac{r^2}{R^2}\right]}{s}+\frac{c}{\varepsilon\rho}\left\{\frac{6ra_q\left[\frac{t_{\text{ц}}-t_n}{R^2}+\omega\mathrm{Lu}\right]}{s^2}+\right.$$

$$+\left[\frac{\Omega_1\widetilde{P}_1-\Omega_3\widetilde{Q}_1}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}+\frac{\Omega_2'\widetilde{P}_1-\Omega_4\widetilde{Q}_1}{s^2(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\right](1-\nu_2^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r-$$

$$-\left[\frac{\Omega_1\widetilde{P}_2-\Omega_3\widetilde{Q}_2}{s(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}+\frac{\Omega_2'\widetilde{P}_2-\Omega_4\widetilde{Q}_2}{s^2(\widetilde{Q}_2\widetilde{P}_1-\widetilde{Q}_1\widetilde{P}_2)}\right]\times$$

$$\left.\times(1-\nu_1^2)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r\right\}. \tag{2.43}$$

Here,

$$\Omega_2'=-6a_q\left[K_1\left(\frac{t_{\text{ц}}-t_n}{R^2}+\omega\,\mathrm{Lu}\right)+\omega\mathrm{Lu}+\frac{b}{6a_q}\right],$$

with the values of the remaining quantities given in Section 1d. $L^{-1}[\vartheta_L(r,s)]$, $L^{-1}[u_L(r,s)]$ can be found using the generalized expansion theorem. The inversions of the basic expressions (2.42) and (2.43) were obtained in Section 1d. The final solution of the problem is

$$T_1=\frac{t(r,\tau)-t_n}{t_c^0-t_n}=1-\mathrm{Pd}\left[\mathrm{Fo}-\frac{1}{6}\left(1-\frac{r^2}{R^2}+\frac{2}{\mathrm{Bi}_q}\right)\right]-$$

$$-\sum_{n=1}^{\infty}\left(C_{n1}\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}}-\right.$$

$$-C_{n2}\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}}\Bigg)\exp(-\mu_n^2\mathrm{Fo}),\tag{2.44}$$

$$\Theta_1=\frac{\theta_n-\theta(r,\tau)}{\theta_n-\theta_p}=1+\frac{1}{6}\mathrm{PnPd}\left(1-\frac{r^2}{R^2}\right)-$$

$$-\sum_{n=1}^{\infty}\left[C^*_{n2}(1-\nu_1^2)\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}}-\right.$$

$$\left.-C^*_{n1}(1-\nu_2^2)\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}}\right]\exp(-\mu_n^2\mathrm{Fo}),\tag{2.45}$$

where,

$$C_{nj}=2\frac{G_1P_{nj}-G_2Q_{nj}}{\mu_n\psi_n},\quad C^*_{nj}\doteq\frac{C_{nj}}{\varepsilon\,\mathrm{Ko}},\tag{2.46}$$

$$G_1=1-2W\left(\frac{1}{\mathrm{Bi}_q}-\frac{3}{\mu_n^2}\right)-\varepsilon\mathrm{Ko}K_1-$$

$$-6\varepsilon\mathrm{Ko}\,K_2\frac{(K_1+1)\,\mathrm{Lu}}{\mu_n^2}+\frac{\mathrm{Pd}}{\mu_n^2},\tag{2.47}$$

$$G_2=2\varepsilon\mathrm{Ko}K_2\left(\frac{1}{\mathrm{Bi}_m}-\frac{3\mathrm{Lu}}{\mu_n^2}\right)-\varepsilon\,\mathrm{Ko},\tag{2.48}$$

μ_n is determined from the solutions of the characteristic equation

$$Q_{n2}P_{n1}-Q_{n1}P_{n2}=0.$$

e) Some Considerations on the Method of Determining Heat and Mass Transfer Coefficients.

The solutions (2.14'), (2.15') and (2.29'), (2.30') can be used in developing a method for the determination of heat and mass transfer coefficients of bound matter (see also A.V. Lykov, 1935; E. P. Shurygina, 1941; Yu. A. Mikhaylov, 1956). The essence of the method is as follows: The linear change of temperature of the medium and the temperature of two points in a moist body are recorded starting with $\mathrm{Fo} > \mathrm{Fo}_{\mathrm{quasistationary}}$. Also, the moisture distribution in the material is determined at a given time. The data are used to calculate the following:

1. Predvoditelev number:

$$\mathrm{Pd} = \frac{AR^2}{\xi^2 - R^2} \, \frac{t(R, \tau) - t(\xi, \tau)}{t_c - t_0} ,$$

where A = 2 for a plate, A = 4 for a cylinder and A = 6 for a sphere, ξ is an arbitrary coordinate of the moist body.

2. Thermal diffusivity coefficient:

$$a_q = \frac{b}{A} \, \frac{\xi^2 - R^2}{t(R, \tau) - t(\xi, \tau)} .$$

3. The Posnov number:

$$\mathrm{Pn} = \frac{A(\Theta - 1)}{\mathrm{Pd}} .$$

4. The Soret coefficient:

$$\delta = \mathrm{Pn} \frac{\theta_0 - \theta_p}{t_c - t_0} = \mathrm{Pn}' \, \frac{u_0 - u_p}{t_c - t_0} ,$$

where θ_0 and θ_p are the initial and equilibrium values of the mass transfer potential; u_0 and u_p are the initial and equilibrium moisture content.

5. Volumetric heat capacity:

$$c_q \gamma_0 = \frac{\alpha A}{2 bR} [t_c - t(R, \tau)] .$$

6. Thermal conductivity:

$$\lambda = c_q \gamma_0 a_q .$$

The advantage of this method is that all the indicated quantities can be determined from one experiment. The disadvantage of the method lies in the fact that it is difficult to control the temperature of the experiment to obtain a linear temperature change of the external medium.

3. NON-STATIONARY HEAT AND MASS TRANSFER POTENTIAL FIELDS. TEMPERATURE OF THE EXTERNAL MEDIUM – AN EXPONENTIAL FUNCTION OF TIME.

Let us consider the solution of the system of heat and mass transfer equations (I) to (VI) for the case of exponential dependence of the external temperature on time. The method of solving these equations at $t_c = t_c^0 e^{-\varkappa\tau}$ is essentially the same as in the problems discussed heretofore. Thus, only the final solutions will be given:

a) Solutions for an Infinite Plate. Uniform Initial Conditions

$$T = \frac{t(x, \tau) - t_0}{t_c^0 - t_0} = -W_0 + W_c \times$$

$$\times \frac{\bar{P}_1 \cos\left(\nu_2 \sqrt{\mathrm{Pd}'} \frac{x}{R}\right) - \bar{\bar{P}}_2 \cos\left(\nu_1 \sqrt{\mathrm{Pd}'} \frac{x}{R}\right)}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2} \exp(-\mathrm{Pd}'\mathrm{Fo}) -$$

$$-\sum_{n=1}^{\infty}\left(C_{n2} \cos \nu_1 \mu_n \frac{x}{R} - C_{n1} \cos \nu_2 \mu_n \frac{x}{R}\right) \times$$

$$\times \exp(-\mu_n^2 \mathrm{Fo}), \tag{3.1}$$

$$\Theta = \frac{\theta_0 - \theta(x, \tau)}{\theta_0 - \theta_p} = 1 + \frac{W_c}{\varepsilon \mathrm{Ko}} \times$$

$$\times \frac{\bar{\bar{P}}_2 (1 - \nu_1^2) \cos\left(\nu_1 \sqrt{\mathrm{Pd}'} \frac{x}{R}\right) - \bar{\bar{P}}_1 (1 - \nu_2^2) \cos\left(\nu_2 \sqrt{\mathrm{Pd}'} \frac{x}{R}\right)}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2} \times$$

$$\times \exp(-\mathrm{Pd}'\mathrm{Fo}) - \sum_{n=1}^{\infty}\left[C_{n1}^* (1 - \nu_2^2) \cos \nu_2 \mu_n \frac{x}{R} -\right.$$

$$\left. - C_{n2}^* (1 - \nu_1^2) \cos \nu_1 \mu_n \frac{x}{R}\right] \exp(-\mu_n^2 \mathrm{Fo}), \tag{3.2}$$

where

$$C_{nj} = 2 \frac{\varepsilon \mathrm{Ko} Q_{nj} - \left[\varepsilon \mathrm{Ko} K_1 + W_0 - \frac{W_c}{1 - \mathrm{Pd}'/\mu_n^2}\right] \mathrm{P}_{nj}}{\mu_n \psi_n}$$

$$C_{nj}^* = \frac{C_{nj}}{\varepsilon \mathrm{Ko}}, \tag{3.3}$$

$$\bar{\bar{Q}}_j = [1 + K_1(1 - \nu_j^2)] \cos \nu_j \sqrt{\mathrm{Pd}'} -$$

$$-\frac{1}{\mathrm{Bi}_q} \nu_j \sqrt{\mathrm{Pd}'} \sin \nu_j \sqrt{\mathrm{Pd}'},$$

$$\bar{\bar{P}}_j = (1 - \nu_j^2) \cos \nu_j \sqrt{\mathrm{Pd}'} - [(1 - \nu_j^2) + \mathrm{Fe}] \times \times \frac{\nu_j \sqrt{\mathrm{Pd}'}}{\mathrm{Bi}_m} \sin \nu_j \sqrt{\mathrm{Pd}'} .$$

The average dimensionless transfer potentials are

$$\bar{T} = - W_0 + W_c N \exp(- \mathrm{Pd}'\mathrm{Fo}) - - \sum_{n=1}^{\infty} D_n \exp(-\mu_n^2 \mathrm{Fo}), \tag{3.4}$$

$$\bar{\theta} = 1 + W_c N^* \exp(-\mathrm{Pd}'\mathrm{Fo}) - - \sum_{n=1}^{\infty} D_n^* \exp(-\mu_n^2 \mathrm{Fo}). \tag{3.5}$$

Here,

$$N = \frac{\bar{\bar{P}}_1 \dfrac{\sin(\nu_2 \sqrt{\mathrm{Pd}'})}{\nu_2 \sqrt{\mathrm{Pd}'}} - \bar{\bar{P}}_2 \dfrac{\sin(\nu_1 \sqrt{\mathrm{Pd}'})}{\nu_1 \sqrt{\mathrm{Pd}'}}}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2},$$

$$N^* = \frac{1}{\varepsilon \mathrm{Ko}} \times$$

$$\times \frac{\bar{\bar{P}}_2 (1 - \nu_1^2) \dfrac{\sin(\nu_1 \sqrt{\mathrm{Pd}'})}{\nu_1 \sqrt{\mathrm{Pd}'}} - \bar{\bar{P}}_1 (1 - \nu_2^2) \dfrac{\sin(\nu_2 \sqrt{\mathrm{Pd}'})}{\nu_2 \sqrt{\mathrm{Pd}'}}}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2}$$

The expressions for D_n and D_n^* have the usual form. The local and average dimensionless rates of change of the potentials are

$$\frac{dT}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} \left(C_{n2} \cos \nu_1 \mu_n \frac{x}{R} - - C_{n1} \cos \nu_2 \mu_n \frac{x}{R} \right) \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}) - W_c \mathrm{Pd}' \times$$

$$\times \frac{\bar{\bar{P}}_1 \cos \left(\nu_2 \sqrt{\mathrm{Pd}'} \frac{x}{R} \right) - \bar{\bar{P}}_2 \cos \left(\nu_1 \sqrt{\mathrm{Pd}'} \frac{x}{R} \right)}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2} \exp(-\mathrm{Pd}'\mathrm{Fo}), \tag{3.6}$$

$$\frac{d\bar{T}}{d\mathrm{Fo}} = \sum_{n=1}^{\infty} D_n \mu_n^2 \exp(-\mu_n^2 \mathrm{Fo}) - W_c \mathrm{Pd}' N, \tag{3.7}$$

$$\frac{d\Theta}{d\mathrm{Fo}}=\sum_{n=1}^{\infty}\left[C_{n1}^{*}(1-\nu_2^2)\cos\nu_2\mu_n\frac{x}{R}-C_{n2}^{*}(1-\nu_1^2)\cos\nu_1\mu_n\frac{x}{R}\right]\times$$

$$\times\ \mu_n^2\exp(-\mu_n^2\,\mathrm{Fo})-\frac{W_c}{\varepsilon\mathrm{Ko}}\times$$

$$\times\mathrm{Pd}'\frac{\bar{\bar{P}}_2(1-\nu_1^2)\cos\left(\nu_1\sqrt{\mathrm{Pd}'}\,\frac{x}{R}\right)-\bar{\bar{P}}_1(1-\nu_2^2)\cos\left(\nu_2\sqrt{\mathrm{Pd}'}\,\frac{x}{R}\right)}{\bar{\bar{Q}}_2\bar{\bar{P}}_1-\bar{\bar{Q}}_1\bar{\bar{P}}_2}\times$$

$$\times\exp(-\mathrm{Pd}'\,\mathrm{Fo}), \tag{3.8}$$

$$\frac{d\bar{\Theta}}{d\mathrm{Fo}}=\sum_{n=1}^{\infty}D_n^{*}\,\mu_n^2\exp(-\mu_n^2\,\mathrm{Fo})-W_c\mathrm{Pd}'\,N^{*}. \tag{3.9}$$

In equations (3.1), (3.2), (3.4) and (3.9) the expression $\mathrm{Pd}'=\frac{\varkappa R^2}{a_q}$ is the Predvoditelev number for the problems with an exponential rate of change of temperature with time; $W_c=\frac{t_c^0}{t_c^0-t_0}$, $W_0=\frac{t_0}{t_c^0-t_0}$ are the parametric dimensionless numbers for temperature.

b) Solutions for a Sphere. Uniform Initial Conditions

$$T=\frac{t(r,\tau)-t_0}{t_c^0-t_0}=-W_0+W_c\times$$

$$\times\frac{\bar{\bar{P}}_1\dfrac{\sin\left(\nu_2\sqrt{\mathrm{Pd}'}\,\frac{r}{R}\right)}{\frac{r}{R}}-\bar{\bar{P}}_2\dfrac{\sin\left(\nu_1\sqrt{\mathrm{Pd}'}\,\frac{r}{R}\right)}{\frac{r}{R}}}{\bar{\bar{Q}}_2\bar{\bar{P}}_1-\bar{\bar{Q}}_1\bar{\bar{P}}_2}\exp(-\mathrm{Pd}'\mathrm{Fo})-$$

$$-\sum_{n=1}^{\infty}\left(C_{n1}\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}}-C_{n2}\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}}\right)\exp(-\mu_n^2\mathrm{Fo}), \tag{3.10}$$

$$\Theta=\frac{\theta_0-\theta(r,\tau)}{\theta_0-\theta_p}=1+\frac{W_c}{\varepsilon\mathrm{Ko}}\left(\bar{\bar{Q}}_2\bar{\bar{P}}_2-\bar{\bar{Q}}_1\bar{\bar{P}}_2\right)^{-1}\times$$

$$\times\left[\bar{\bar{P}}_2(1-\nu_1^2)\frac{\sin\left(\nu_1\sqrt{\mathrm{Pd}'}\,\frac{r}{R}\right)}{\frac{r}{R}}-\right.$$

$$-\overline{P}_1(1-\nu_2^2)\,\frac{\sin\left(\nu_2\sqrt{\mathrm{Pd}'}\,\frac{r}{R}\right)}{\frac{r}{R}}\Bigg]\exp(-\mathrm{Pd}'\mathrm{Fo})-$$

$$-\sum_{n=1}^{\infty}\left[C_{n2}^{*}(1-\nu_1^2)\,\frac{\sin\nu_1\mu_n\frac{r}{R}}{\frac{r}{R}}-\right.$$

$$\left.-\,C_{n1}^{*}\,(1-\nu_2^2)\,\frac{\sin\nu_2\mu_n\frac{r}{R}}{\frac{r}{R}}\right]\exp(-\mu_n^2\,\mathrm{Fo}). \qquad (3.11)$$

Here, C_{nj} and C_{nj}^{*} are determined as in equations (3.3); other terms are determined as in the previous solutions for a sphere; and

$$\overline{Q}_j=\frac{1}{\mathrm{Bi}_q}\nu_j\sqrt{\mathrm{Pd}'}\cos(\nu_j\sqrt{\mathrm{Pd}'})+$$

$$+\left[1-\frac{1}{\mathrm{Bi}_q}+K_1(1-\nu_j^2)\right]\sin(\nu_j\sqrt{\mathrm{Pd}'}),$$

$$P_j=\left[(1-\nu_j^2)+\mathrm{Fe}\right]\frac{1}{\mathrm{Bi}_m}\nu_j\sqrt{\mathrm{Pd}'}\cos\nu_j\sqrt{\mathrm{Pd}'}+$$

$$+\left[(1-\nu_j^2)-\frac{(1-\nu_j^2)+\mathrm{Fe}}{\mathrm{Bi}_m}\right]\sin(\nu_j\sqrt{\mathrm{Pd}'}).$$

As usual, the characteristic roots, μ_n, are determined by

$$Q_{n2}P_{n1}-Q_{n1}P_{n2}=0.$$

The average dimensionless temperature and the dimensionless mass potential are determined by the following:

$$\overline{T}=-W_0+W_cN\exp(-\mathrm{Pd}'\mathrm{Fo})-\sum_{n=1}^{\infty}D_n\exp(-\mu_n^2\mathrm{Fo}) \qquad (3.12)$$

$$\overline{\Theta}=1+W_cN^{*}\exp(-\mathrm{Pd}'\mathrm{Fo})-\sum_{n=1}^{\infty}D_n^{*}\exp(-\mu_n^2\mathrm{Fo}), \qquad (3.13)$$

where

$$N=\frac{\bar{\bar{P}}_1\xi_2-\bar{\bar{P}}_2\xi_1}{\bar{\bar{Q}}_2\bar{\bar{P}}_1-\bar{\bar{Q}}_1\bar{\bar{P}}_2},$$

$$N^*=\frac{1}{\varepsilon\,\text{Ko}}\;\frac{(1-\nu_1^2)\xi_1\bar{\bar{P}}_2-(1-\nu_2^2)\,\xi_2\bar{\bar{P}}_1}{\bar{\bar{Q}}_2\bar{\bar{P}}_1-\bar{\bar{Q}}_1\bar{\bar{P}}_2},$$

$$\xi_j=\frac{3\left[\sin\left(\nu_j\sqrt{\text{Pd}'}\right)-\nu_j\sqrt{\text{Pd}'}\cos\left(\nu_j\sqrt{\text{Pd}'}\right)\right]}{\left(\nu_j\sqrt{\text{Pd}'}\right)^2},$$

$$D_n=D_{n1}-D_{n2},\; D_n^*=D_{n2}^*-D_{n1}^*.$$

Differentiation of equations (3.10) to (3.13) with respect to the dimensionless time yields solutions for the rates of heating and drying of the moist material.

c) Solutions for an Infinite Plate. Parabolic Initial Conditions.

$$T_1=\frac{t(x,\tau)-t_{\text{п}}}{t_c^0-t_{\text{п}}}=-W_{\text{пп}}+$$

$$+W_{\text{cп}}\frac{\bar{\bar{P}}_1\cos\left(\nu_2\sqrt{\text{Pd}'}\,\frac{x}{R}\right)-\bar{\bar{P}}_2\cos\left(\nu_1\sqrt{\text{Pd}'}\,\frac{x}{R}\right)}{\bar{\bar{Q}}_2\bar{\bar{P}}_1-\bar{\bar{Q}}_1\bar{\bar{P}}_2}\times$$

$$\times\exp\left(-\text{Pd}'\,\text{Fo}\right)-\sum_{n=1}^{\infty}\left(C_{n2}\cos\nu_1\mu_n\frac{x}{R}-\right.$$

$$\left.-C_{n1}\cos\nu_2\mu_n\frac{x}{R}\right)\exp\left(-\mu_n^2\,\text{Fo}\right), \tag{3.14}$$

$$\Theta_1=\frac{\theta_{\text{п}}-\theta(x,\tau)}{\theta_{\text{п}}-\theta_p}=1+\frac{W_{\text{cп}}}{\varepsilon\,\text{Ko}}\times$$

$$\times\left[\bar{\bar{P}}_2(1-\nu_1^2)\cos\left(\nu_1\sqrt{\text{Pd}'}\,\frac{x}{R}\right)-\right.$$

$$\left.-\bar{\bar{P}}_1(1-\nu_2^2)\cos\left(\nu_2\sqrt{\text{Pd}'}\,\frac{x}{R}\right)\right]\frac{\exp\left(-\text{Pd}'\,\text{Fo}\right)}{\bar{\bar{Q}}_2\bar{\bar{P}}_1-Q_1\bar{\bar{P}}_2}-$$

$$- \sum_{n=1}^{\infty} \left[C_{n1}^{*} (1 - \nu_2^2) \cos \nu_2 \mu_n \frac{x}{R} - C_{n2}^{*} (1 - \nu_1^2) \cos \nu_1 \mu_n \frac{x}{R} \right] \times$$

$$\times \exp(-\mu_n^2 \mathrm{Fo}), \tag{3.15}$$

where,

$$C_{nj} = 2 \frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \psi_n}, \quad C_{nj}^{*} = \frac{C_{nj}}{\varepsilon \mathrm{Ko}}, \tag{3.16}$$

$$G_1 = 1 - 2W \left(\frac{1}{\mathrm{Bi}_q} - \frac{1}{\mu_n^2} \right) - \varepsilon \mathrm{Ko} K_1 -$$

$$- 2 \varepsilon \mathrm{Ko} K_2 \frac{(K_1 + 1) \mathrm{Lu}}{\mu_n^2} - W_{\text{cп}} + \frac{W_{\text{cп}}}{1 - \mathrm{Pd}'/\mu_n^2},$$

$$G_2 = 2 \varepsilon \mathrm{Ko} K_2 \left(\frac{1}{\mathrm{Bi}_m} - \frac{\mathrm{Lu}}{\mu_n^2} \right) - \varepsilon \mathrm{Ko}.$$

The nomenclature for the average values is the same as employed in the problem with uniform initial conditions. It should be noted that instead of W_0 and W_c, we employ here $W_{\text{пп}}$ and $W_{\text{cп}}$, respectively. The difference, however, consists in the values of C_{nj} and C_{nj}^{*}. The latter are determined by equation (3.16), and

$$W_{\text{пп}} = W_{\text{cп}} - 1 = \frac{t_{\text{п}}}{t_c^0 - t_{\text{п}}}.$$

d) Solutions for a Sphere. Parabolic Initial Conditions.

$$T_1 = \frac{t(r, \tau) - t_{\text{п}}}{t_c^0 - t_{\text{п}}} = - W_{\text{пп}} +$$

$$+ W_{\text{cп}} \frac{\bar{\bar{P}}_1 \dfrac{\sin(\nu_2 \sqrt{\mathrm{Pd}'} r/R)}{r/R} - \bar{\bar{P}}_2 \dfrac{\sin(\nu_1 \sqrt{\mathrm{Pd}'} r/R)}{r/R}}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2} \times$$

$$\times \exp(-\mathrm{Pd}' \mathrm{Fo}) - \sum_{n=1}^{\infty} \left(C_{n1} \frac{\sin \nu_2 \mu_n r/R}{r/R} - \right.$$

$$\left. - C_{n2} \frac{\sin \nu_1 \mu_n r/R}{r/R} \right) \exp(-\mu_n^2 \mathrm{Fo}), \tag{3.17}$$

$$\Theta_1 = \frac{\theta_{\text{п}} - \theta(r, \tau)}{\theta_{\text{п}} - \theta_p} = 1 + \frac{W_{\text{cп}}}{\varepsilon \mathrm{Ko}} \times$$

$$\times\left[\bar{\bar{P}}_2\left(1-\nu_1^2\right)\frac{\sin\left(\nu_1\sqrt{\mathrm{Pd}'}\,r/R\right)}{r/R}-\right.$$

$$\left.-\bar{\bar{P}}_1\left(1-\nu_2^2\right)\frac{\sin\left(\nu_2\sqrt{\mathrm{Pd}'}\,r/R\right)}{r/R}\right]\frac{\exp\left(-\mathrm{Pd}'\,\mathrm{Fo}\right)}{\bar{\bar{Q}}_2\bar{\bar{P}}_1-\bar{\bar{Q}}_1\bar{\bar{P}}_2}-$$

$$-\sum_{n=1}^{\infty}\left[C_{n2}^{*}\left(1-\nu_1^2\right)\frac{\sin\nu_1\mu_n r/R}{r/R}-\right.$$

$$\left.-C_{n1}^{*}\left(1-\nu_2^2\right)\frac{\sin\nu_2\mu_n r/R}{r/R}\right]\exp\left(-\mu_n^2\,\mathrm{Fo}\right), \qquad (3.18)$$

where,

$$C_{nj}=2\,\frac{G_1P_{nj}-G_2Q_{nj}}{\mu_n\,\psi_n},\quad C_{nj}^{*}=\frac{C_{nj}}{\varepsilon\,\mathrm{Ko}},$$

$$G_1=1-2\,W\left(\frac{1}{\mathrm{Bi}_q}-\frac{3}{\mu_n^2}\right)-\varepsilon\,\mathrm{Ko}\,K_1-$$

$$-G\,\varepsilon\,\mathrm{Ko}\,K_2\,\frac{(K_1+1)\,\mathrm{Lu}}{\mu_n^2}-$$

$$-W_{\mathrm{cn}}+\frac{W_{\mathrm{cn}}}{1-\mathrm{Pd}'/\mu_n^2},$$

$$G_2=2\,\varepsilon\,\mathrm{Ko}\,K_2\left(\frac{1}{\mathrm{Bi}_m}-\frac{3\,\mathrm{Lu}}{\mu_n^2}\right)\varepsilon\,\mathrm{Ko}.$$

4. GENERALIZED SOLUTIONS OF HEAT AND MASS TRANSFER EQUATIONS

A comparison of solutions presented in Sections 1-3 for bodies of the same configuration shows that they contain a number of common elements (for example, ψ_n, Q_{nj}, P_{nj}, A_{nj}, B_{nj}, characteristic roots, etc.). Therefore, the final solutions can be presented in a form applicable both to problems with various initial conditions and problems with various modes of temperature change of the external medium. Such generalized solutions are presented below for an infinite plate and a sphere.

The generalized solution of heat and mass transfer in an infinite plate can be presented in the following form:

$$T = 1 + \Pi - \sum_{n=1}^{\infty} \left(C_{n2} \cos \nu_1 \mu_n x/R - C_{n1} \cos \nu_2 \mu_n x/R\right) \times$$

$$\times \exp\left(-\mu_n^2 \, \mathrm{Fo}\right),$$

$$\Theta = 1 + \Pi^* - \sum_{n=1}^{\infty} \left[C_{n1}^* \left(1 - \nu_2^2\right) \cos \nu_2 \mu_n x/R - \right.$$

$$\left. - C_{n2}^* \left(1 - \nu_1^2\right) \cos \nu_1 \mu_n x/R\right] \exp\left(-\mu_n^2 \, \mathrm{Fo}\right),$$

where

a — for uniform initial conditions

$$C_{nj} = 2 \, \frac{P_{nj} + \varepsilon \, \mathrm{Ko} \left(Q_{nj} - K_1 P_{nj}\right) + M P_{nj}}{\mu_n \psi_n},$$

$$C_{nj}^* = \frac{1}{\varepsilon \, \mathrm{Ko}} \, C_{nj},$$

$$t(x, 0) = t_0, \quad \theta(x, 0) = \theta_0;$$

b — for parabolic initial conditions

$$C_{nj} = 2 \, \frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \psi_n}, \quad C_{nj}^* = \frac{1}{\varepsilon \, \mathrm{Ko}} \, C_{nj},$$

$$t(x, 0) = t_{\text{п}} \qquad \theta(x, 0) = \theta_{\text{п}}.$$

Here,

$$\nu_j^2 = \frac{1}{2} \left[\left(1 + \varepsilon \, \mathrm{KoPn} + \frac{1}{\mathrm{Lu}}\right) + (-1)^j \times \right.$$

$$\left. \times \sqrt{\left(1 + \varepsilon \, \mathrm{KoPn} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}}\right],$$

$$\psi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2},$$

$$A_{nj} = \left[1 + \frac{1}{\mathrm{Bi}_q} + \left(1 - \nu_j^2\right) K_1\right] \sin \nu_j \mu_n +$$

$$+ \frac{\nu_j \mu_n}{\mathrm{Bi}_q} \cos \nu_j \mu_n,$$

$$B_{nj} = \left(1 - \nu_j^2\right) \sin \nu_j \mu_n +$$

$$+\frac{(1-\nu_j^2)+\varepsilon\,\mathrm{KoPn}}{\mathrm{Bi}_m}\,(\sin\nu_j\mu_n+\nu_j\mu_n\cos\nu_j\mu_n),$$

$$Q_{nj}=[1+(1-\nu_j^2)\,K_1]\cos\nu_j\mu_n-\frac{\nu_j\mu_n}{\mathrm{Bi}_q}\sin\nu_j\mu_n,$$

$$P_{nj}=(1-\nu_j^2)\cos\nu_j\mu_n-[(1-\nu_j^2)+\varepsilon\,\mathrm{KoPn}]\times$$

$$\times\frac{\nu_j\mu_n}{\mathrm{Bi}_m}\sin\nu_j\mu_n,$$

$$G_1=1-2W\left(\frac{1}{\mathrm{Bi}_q}-\frac{1}{\mu_n^2}\right)-\varepsilon\,\mathrm{Ko}K_1-$$

$$-2\varepsilon\,\mathrm{Ko}K_2\,\frac{(K_1+1)\,\mathrm{Lu}}{\mu_n^2}+M,$$

$$G_2=2\,\varepsilon\,\mathrm{Ko}K_2\left(\frac{1}{\mathrm{Bi}_m}-\frac{\mathrm{Lu}}{\mu_n^2}\right)-\varepsilon\,\mathrm{Ko},$$

The μ_n are determined by solutions of $Q_2P_1=Q_1P_2$.

For t_c = constant,

$$\Pi=0,\ \Pi^*=0,\ M=0,$$

$$t_c=t_c^0-b\,\tau$$

$$\Pi=-\,\mathrm{Pd}\left[\mathrm{Fo}-\frac{1}{2}\left(1-\frac{x^2}{R^2}+\frac{2}{\mathrm{Bi}_q}\right)\right],$$

$$\Pi^*=\frac{1}{2}\,\mathrm{PnPd}\left(1-\frac{x^2}{R^2}\right),\quad M=\frac{\mathrm{Pd}}{\mu_n^2},$$

$$t_c=t_c^0\,e^{-\varkappa\tau}$$

$$\Pi=-W_c+W_c\,\frac{\bar{\bar{P}}_1\cos\left(\nu_2\sqrt{\mathrm{Pd}'}\,\frac{x}{R}\right)-\bar{\bar{P}}_2\cos\left(\nu_1\sqrt{\mathrm{Pd}'}\,\frac{x}{R}\right)}{\bar{\bar{Q}}_2\bar{\bar{P}}_1-\bar{\bar{Q}}_1\bar{\bar{P}}_2}\times$$

$$\times\exp\,(-\,\mathrm{Pd}'\mathrm{Fo}),$$

$$\Pi^*=\frac{W_c}{\varepsilon\,\mathrm{Ko}}\times$$

$$\times\frac{\bar{\bar{P}}_2(1-\nu_1^2)\cos\left(\nu_1\sqrt{\mathrm{Pd}'}\,\frac{x}{R}\right)-\bar{\bar{P}}_1\,(1-\nu_2^2)\cos\left(\nu_2\sqrt{\mathrm{Pd}'}\,\frac{x}{R}\right)}{\bar{\bar{Q}}_2\bar{\bar{P}}_1-\bar{\bar{Q}}_1\bar{\bar{P}}_2}\times$$

$$\times\exp\,(-\,\mathrm{Pd}'\mathrm{Fo}),$$

$$M = -W_c + \frac{W_c}{1 - \dfrac{\mathrm{Pd}'}{\mu_n^2}},$$

$$\bar{\bar{Q}}_j = [1 + K_1 (1 - \nu_j^2)] \cos (\nu_j \sqrt{\mathrm{Pd}'}) - $$

$$- \frac{\nu_j \sqrt{\mathrm{Pd}'}}{\mathrm{Bi}_q} \sin (\nu_j \sqrt{\mathrm{Pd}'}),$$

$$\bar{\bar{P}}_j = (1 - \nu_j^2) \cos (\nu_j \sqrt{\mathrm{Pd}'}) -$$

$$- [(1 - \nu_j^2) + \varepsilon \mathrm{KoPn}] \frac{\nu_j \sqrt{\mathrm{Pd}'}}{\mathrm{Bi}_m} \sin (\nu_j \sqrt{\mathrm{Pd}'}).$$

The generalized solution for internal heat and mass transfer in a sphere is

$$T = 1 + \Pi - \sum_{n=1}^{\infty} \left(C_{n1} \frac{\sin \nu_2 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}} - \right.$$

$$\left. - C_{n2} \frac{\sin \nu_1 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}} \right) \exp (-\mu_n^2 \mathrm{Fo}),$$

$$\theta = 1 + \Pi^* - \sum_{n=1}^{\infty} \left[C_{n2}^* (1 - \nu_1^2) \frac{\sin \nu_1 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}} - \right.$$

$$\left. - C_{n1}^* (1 - \nu_2^2) \frac{\sin \nu_2 \mu_n \dfrac{r}{R}}{\dfrac{r}{R}} \right] \exp (-\mu_n^2 \mathrm{Fo}),$$

where

a — for uniform initial conditions

$$C_{nj} = 2 \frac{P_{nj} + \varepsilon \mathrm{Ko} (Q_{nj} - K_1 P_{nj}) + M P_{nj}}{\mu_n \psi_n}, \quad C_{nj}^* = \frac{1}{\varepsilon \mathrm{Ko}} C_{nj},$$

$$t (r, 0) = t_0, \quad \theta (r, 0) = \theta_0;$$

b — for parabolic initial conditions

$$C_{nj} = 2 \frac{G_1 P_{nj} - G_2 Q_{nj}}{\mu_n \psi_n}, \quad C_{nj}^* = \frac{1}{\varepsilon \mathrm{Ko}} C_{nj},$$

$$t (r, 0) = t_{\text{п}}, \quad \theta (r, 0) = \theta_{\text{п}}.$$

Here

$$\nu_j^2 = \frac{1}{2}\left[\left(1 + \varepsilon\,\mathrm{KoPn} + \frac{1}{\mathrm{Lu}}\right) + (-1)^j \sqrt{\left(1 + \varepsilon\,\mathrm{KoPn} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}}\,\right],$$

$$\psi_n = \nu_1 A_{n1} P_{n2} + \nu_2 B_{n2} Q_{n1} - \nu_2 A_{n2} P_{n1} - \nu_1 B_{n1} Q_{n2},$$

$$A_{nj} = [1 + K_1\,(1 - \nu_j^2)]\,\cos \nu_j \mu_n - \frac{\nu_j \mu_n}{\mathrm{Bi}_q}\,\sin \nu_j \mu_n\,,$$

$$B_{nj} = (1 - \nu_j^2)\,\cos \nu_j \mu_n - [(1 - \nu_j^2) + \varepsilon\,\mathrm{KoPn}]\,\frac{\nu_j \mu_n}{\mathrm{Bi}_m}\,\sin \nu_j \mu_n\,,$$

$$Q_{nj} = \left[\,1 - \frac{1}{\mathrm{Bi}_q} + K_1 (1 - \nu_j^2)\,\right] \sin \nu_j \mu_n + \frac{\nu_j \mu_n}{\mathrm{Bi}_q}\,\cos \nu_j \mu_n\,,$$

$$P_{nj} = \left[\,(1 - \nu_j^2) - \frac{(1 - \nu_j^2) + \varepsilon\,\mathrm{KoPn}}{\mathrm{Bi}_m}\,\right] \sin \nu_j \mu_n + [(1 - \nu_j^2) + \varepsilon\,\mathrm{Ko\,Pn}]\,\frac{\nu_j \mu_n}{\mathrm{Bi}_m}\,\cos \nu_j \mu_n\,,$$

$$G_1 = 1 - 2\,W\left(\frac{1}{\mathrm{Bi}_q} - \frac{3}{\mu_n^2}\right) - \varepsilon\,\mathrm{Ko}K_1 - 6\,\varepsilon\,\mathrm{Ko}K_2\,\frac{(K_1 + 1)\,\mathrm{Lu}}{\mu_n^2} + M,$$

$$G_2 = 2\,\varepsilon\,\mathrm{Ko}K_2\left(\frac{1}{\mathrm{Bi}_m} - \frac{3\,\mathrm{Lu}}{\mu_n^2}\right) - \varepsilon\mathrm{Ko},$$

μ_n is determined by the solution of the equation

$$Q_2 P_1 = Q_1 P_2.$$

For

$t_c = \mathrm{const}$ $\quad \Pi = 0, \quad \Pi^* = 0, \quad M = 0,$

$t_c = t_c^0 - b\,\tau$

$$\Pi = -\,\mathrm{Pd}\left[\mathrm{Fo} - \frac{1}{6}\left(1 - \frac{r^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right)\right],$$

$$\Pi^* = \frac{1}{6}\,\mathrm{Pn\,Pd}\,\left(1 - \frac{r^2}{R^2}\right),$$

$$M = \frac{\mathrm{Pd}}{\mu_n^2},$$

$$t_c = t_c^o e^{-\varkappa t}$$

$$\Pi = -W_c + W_c \frac{\bar{\bar{P}}_1 \dfrac{\sin\left(\nu_2 \sqrt{\mathrm{Pd}'} \dfrac{r}{R}\right)}{\dfrac{r}{R}} - \bar{\bar{P}}_2 \dfrac{\sin\left(\nu_1 \sqrt{\mathrm{Pd}'} \dfrac{r}{R}\right)}{\dfrac{r}{R}}}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2} \times$$

$$\times \exp(-\mathrm{Pd}'\mathrm{Fo}),$$

$$\Pi^* = \frac{W_c}{\varepsilon \mathrm{Ko}} \left[\bar{\bar{P}}_2 (1 - \nu_1^2) \frac{\sin\left(\nu_1 \sqrt{\mathrm{Pd}'} \dfrac{r}{R}\right)}{\dfrac{r}{\mathrm{R}}} - \right.$$

$$\left. - \bar{\bar{P}}(1 - \nu_2^2) \frac{\sin\left(\nu_2 \sqrt{\mathrm{Pd}'} \dfrac{r}{R}\right)}{\dfrac{r}{R}} \right] \frac{\exp(-\mathrm{Pd}'\mathrm{Fo})}{\bar{\bar{Q}}_2 \bar{\bar{P}}_1 - \bar{\bar{Q}}_1 \bar{\bar{P}}_2},$$

$$M = -W_c + \frac{W_c}{1 - \dfrac{\mathrm{Pd}'}{\mu_n^2}},$$

$$\bar{\bar{Q}}_j = \left[1 - \frac{1}{\mathrm{Bi}_q} + K_1 (1 - \nu_j^2)\right] \sin(\nu_j \sqrt{\mathrm{Pd}'}) -$$

$$- \frac{\nu_j \sqrt{\mathrm{Pd}'}}{\mathrm{Bi}_q} \cos(\nu_j \sqrt{\mathrm{Pd}'}),$$

$$\bar{\bar{P}}_j = \left[(1 - \nu_j^2) - \frac{(1 - \nu_j^2) + \varepsilon \mathrm{KoPn}}{\mathrm{Bi}_m}\right] \sin(\nu_j \sqrt{\mathrm{Pd}'}) +$$

$$+ [(1 - \nu_j^2) + \varepsilon \mathrm{KoPn}] \frac{\nu_j \sqrt{\mathrm{Pd}'}}{\mathrm{Bi}_m} \cos(\nu_j \sqrt{\mathrm{Pd}'}).$$

5. SPECIAL CASES OF HEAT AND MASS TRANSFER. CERTAIN PROBLEMS IN OBTAINING LIMITING FORMS OF THE SOLUTIONS OF THE SYSTEM OF HEAT AND MASS TRANSFER EQUATIONS.

The non-stationary fields of heat and mass transfer potential for certain special cases can be most conveniently obtained from the solutions of the system of equations (I-VI) by finding limiting forms of the complete solutions. However, the usual methods of finding the mathematical limits of the complete solutions, equations (I-IV), yields results which do not agree with solutions obtained from the incomplete system of differential equations ($\delta = 0$, $\varepsilon = 0$, etc.). Investigations show that in order to obtain correct mathematical limits it is extremely important to consider the form of the resulting characteristic equation and the "form" of the resulting roots. Thus, for an infinite plate when δ approaches zero (which is the same as Pn or Ko approaching zero) the characteristic equation (1.20) has the following form:

$$\left(\cos\mu - \frac{1}{\mathrm{Bi}_q}\mu\sin\mu\right)\left(\cos\frac{\mu}{\sqrt{\mathrm{Lu}}} - \frac{1}{\mathrm{Bi}_m}\frac{\mu}{\sqrt{\mathrm{Lu}}}\times \right.$$
$$\left.\times\sin\frac{\mu}{\sqrt{\mathrm{Lu}}}\right) = 0, \tag{5.1'}$$

i.e. the previous singularity corresponding to the root μ_{no}, contained in the solution of the complete system of equations, has assumed a double form: μ and $\mu^* = \frac{\mu}{\sqrt{\mathrm{Lu}}}$. The limiting form of solution will be correct when the infinite summation of the general solution is repeated a number of times equal to the number of "forms" of the root present in the limiting form of the characteristic equations. For example, the infinite sum of equation (1.28) has to be repeated twice, by substituting for μ_{no}, first μ, and then μ^*. This substitution has to be performed because the characteristic equation (5.1') contains two "forms" of the root. In this way, the limiting forms can be achieved by the usual method. Limiting solutions permit the checking of the correctness of solutions of the complete system of equations. They also permit an easy and rapid method of obtaining solutions of special cases from one general solution; and finally they permit the explanation of new and interesting characteristics and properties of the process. The development of the limiting solutions for the system of heat and mass transfer equations has disclosed a new physical significance of the $\sqrt{\mathrm{Lu}}$ number, which can now be considered a measure of interaction between heat and mass transfer. The method has also shown the possibility, from a formal mathematical point of view, of calculating heat and mass transfer in incomplete systems ($\delta = 0$, $\varepsilon = 0$) in terms of either mass transfer or heat transfer quantities. It is of interest to note that the $\sqrt{\mathrm{Lu}}$ in heat and mass transfer processes fulfills the same role as the square of the velocity of light in the well-known Einstein equation: $E = c^2 m$, which determines the interaction between the quantities of energy and mass.

By means of the above method of obtaining limiting solutions we can obtain solutions for special cases of heat and mass transfer in moist materials.

a) Absence of Thermal Mass Transfer ($\delta = 0$, $\mathrm{Pn} = 0$). The solution for heat and mass transfer in an infinite plate with uniform initial conditions can be written as follows:

$$T = \frac{t(x,\tau) - t_0}{t_c - t_0} = 1 - \sum_{n=1}^{\infty} A_n \left(1 + \frac{\varphi_n}{\psi_n}\right) \cos \mu_n \frac{x}{R} \times$$

$$\times \exp\left(-\mu_n^2 \mathrm{Fo}\right) - \sum_{n=1}^{\infty} A_n^* \mathrm{Ko} \left(\frac{\varepsilon \mathrm{Lu}}{1 - \mathrm{Lu}} \cos \mu_n^* \frac{x}{R} +\right.$$

$$\left. + \frac{\varphi_n^*}{\psi_n^*} \cos \sqrt{\mathrm{Lu}}\, \mu_n^* \frac{x}{R}\right) \exp\left(-\mu_n^{*2} \mathrm{Fo}^*\right),$$

$$\theta = \frac{\theta_0 - \theta(x,\tau)}{\theta_0 - \theta_p} = 1 - \sum_{n=1}^{\infty} A_n^* \cos \mu_n^* \frac{x}{R} \exp\left(-\mu_n^{*2} \mathrm{Fo}^*\right), \tag{5.1}$$

where

$$\varphi_n = \mathrm{Ko}\left[\frac{\mathrm{Bi}_m}{\mathrm{Bi}_q} \left(1 - \varepsilon - \frac{\varepsilon}{\mathrm{Lu} - 1}\right) \sqrt{\mathrm{Lu}}\, \mu_n \sin \frac{\mu_n}{\sqrt{\mathrm{Lu}}} + \right.$$

$$\left. + \frac{\varepsilon \mathrm{Lu} \mathrm{Bi}_m}{\mathrm{Lu} - 1} \cos \frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right],$$

$$\varphi_n^* = \mathrm{Lu}\left[\left(1 - \varepsilon - \frac{\varepsilon}{\mathrm{Lu} - 1}\right) \mu_n^* \sin \mu_n^* + \frac{\varepsilon \mathrm{Bi}_q}{\mathrm{Lu} - 1} \cos \mu_n^*\right],$$

$$\psi_n = \mathrm{Bi}_m \cos \frac{\mu_n}{\sqrt{\mathrm{Lu}}} - \frac{\mu_n}{\sqrt{\mathrm{Lu}}} \sin \frac{\mu_n}{\sqrt{\mathrm{Lu}}},$$

$$\psi_n^* = \mathrm{Bi}_q \cos \sqrt{\mathrm{Lu}}\, \mu_n^* - \sqrt{\mathrm{Lu}}\, \mu_n^* \sin \sqrt{\mathrm{Lu}}\, \mu_n^*,$$

μ_n and μ_n^* are roots of the characteristic equations

$$\operatorname{ctg} \mu = \frac{1}{\mathrm{Bi}_q} \mu \text{ and } \operatorname{ctg} \mu^* = \frac{1}{\mathrm{Bi}_m} \mu^*, \tag{5.2}$$

and the coefficients A_n and A_n^* are determined by

$$A_n = \frac{2 \sin \mu_n}{\mu_n + \sin \mu_n \cos \mu_n}, \quad A_n^* = \frac{2 \sin \mu_n^*}{\mu_n^* + \sin \mu_n^* \cos \mu_n^*}. \tag{5.3}$$

The values of characteristic roots are given in Table 23 and the values of the coefficients A_n and A_n^* in Table 24. The same data are also presented in Figures 25 and 26.

Table 23. Roots of characteristic equation

$$\operatorname{ctg}\mu = \frac{1}{\mathrm{Bi}_q}\mu \quad \text{and} \quad \operatorname{ctg}\mu^* = \frac{1}{\mathrm{Bi}_m}\mu^*$$

Bi_q Bi_m	μ_1 μ_1^*	μ_2 μ_2^*	μ_3 μ_3^*	μ_4 μ_4^*	μ_5 μ_5^*	μ_6 μ_6^*
0	0,0000	3,1416	6,2832	9,4248	12,5664	15,7080
0,001	0,0316	3,1419	6,2833	9,4249	12,5665	15,7080
0,002	0,0447	3,1422	6,2835	9,4250	12,5665	15,7081
0,004	0,0632	3,1429	6,2838	9,4252	12,5667	15,7082
0,006	0,0774	3,1435	6,2841	9,4254	12,5668	15,7083
0,008	0,0893	3,1441	6,2845	9,4256	12,5670	15,7085
0,010	0,0998	3,1448	6,2848	9,4258	12,5672	15,7086
0,020	0,1410	3.1479	6,2864	9,4269	12,5680	15,7092
0,040	0,1987	3,1543	6,2895	9,4290	12,5696	15,7105
0,060	0,2425	3,1606	6,2927	9,4311	12,5711	15,7118
0,080	0,2791	3,1668	6,2959	9,4333	12,5727	15,7131
0,100	0,3111	3,1731	6,2991	9.4354	12,5743	15,7143
0,200	0,4328	3,2039	6,3148	9,4459	12,5823	15,7207
0,300	0,5218	3,2341	6,3305	9,4565	12,5902	15,7270
0,400	0,5932	3.2636	6,3461	9,4670	12,5981	15,7334
0,500	0,6533	3,2923	6.3616	9,4775	12,6060	15,7397
0,600	0,7061	3,3204	6,3770	9,4879	12,6139	15,7460
0,700	0,7506	3,3477	6,3923	9,4983	12,6218	15,7524
0,800	0,7910	3,3744	6,4074	9,5087	12,6296	15,7587
0,900	0,8274	3,4003	6,4224	9,5190	12,6375	15,7650
1,000	0,8603	3,4256	6,4373	9,5293	12.6453	15,7713
1,500	0,9882	3,5422	6,5097	9,5801	12,6841	15,8026
2,000	1,0769	3,6436	6,5783	9,6296	12,7223	15,8336
3,000	1,1925	3,8088	6,7040	9,7240	12,7966	15,8945
4,000	1,2646	3,9352	6,8140	9,8119	12,8678	15,9536
5,000	1,3138	4,0336	6,9096	9,8928	12,9352	16,0107
6,000	1,3496	4,1116	6,9924	9,9667	12,9988	16,0654
7,000	1,3766	4,1746	7,0640	10,0339	13,0584	16,1177
8,000	1,3978	4,2264	7,1263	10,0949	13,1141	16,1675
9,000	1,4149	4,4694	7,1806	10,1502	13,1660	16,2147
10,000	1,4289	4,3058	7,2281	10,2003	13,2142	16,2594
15,000	1,4729	4,4255	7,3959	10,3898	13,4078	16,4474
20.000	1,4961	4,4915	7,4954	10,5117	13,5420	16,5864
30,000	1,5202	4,5615	7,6057	10,6543	13,7085	16,7691
40,000	1,5325	4,5979	7,6647	10,7334	13,8048	16,8794
50,000	1,5400	4,6202	7,7012	10,7832	13,9666	16,9519
60,000	1,5451	4,6353	7,7259	10,8172	13,9094	17,0026
80,000	1,5514	4,6543	7,7573	10,8606	13,9644	17,0686
100,000	1,5552	4,6658	7,7764	10,8871	13,9981	17,1093
∞	1,5708	4,7124	7,8540	10,9956	14,1372	17,2788

Table 24. Values of the coefficients A_n and A_n^* for an infinite plate.

Bi_q	A_1	A_2	A_3	A_4	A_5	A_6
Bi_m	A_1^*	A_2^*	A_3^*	A_4^*	A_5^*	A_6^*
0,000	1,0000	—0,0000	0,0000	—0,0000	0,0000	—0,0000
0,001	1,0002	—0,0002	0,0000	—0,0000	0,0000	—0,0000
0,002	1,0004	—0,0004	0,0001	—0,0000	0,0000	—0,0000
0,004	1,0008	—0,0008	0,0002	—0,0001	0,0000	—0,0000
0,006	1,0012	—0,0012	0,0003	—0,0001	0,0001	—0,0000
0,008	1,0015	—0,0016	0,0004	—0,0002	0,0001	—0,0001
0,010	1,0020	—0,0020	0,0005	—0,0002	0,0001	—0,0001
0,020	1,0030	—0,0040	0,0010	—0,0004	0,0003	—0,0002
0,040	1,0065	—0,0080	0,0020	—0,0009	0,0005	—0,0003
0,060	1,0099	—0,0119	0,0030	—0,0013	0,0007	—0,0004
0,080	1,0130	—0,0158	0,0040	—0,0018	0,0010	—0,0006
0,100	1,0159	—0,0197	0,0050	—0,0022	0,0013	—0,0008
0,200	1,0312	—0,0381	0,0100	—0,0045	0,0025	—0,0016
0,300	1,0450	—0,0555	0,0148	—0,0067	0,0038	—0,0024
0,400	1,0581	—0,0719	0,0196	—0,0089	9,0050	—0,0032
0,500	1,0701	—0,0873	0,0243	—0,0110	0,0063	—0,0040
0,600	1,0813	—0,1025	0,0289	—0,0132	0,0075	—0,0048
0,700	1,0918	—0,1154	0,0335	—0,0153	0,0087	—0,0056
0,800	1,1016	—0,1282	0,0379	—0,0175	0,0100	—0,0064
0,900	1,1107	—0,1403	0,0423	—0,0196	0,0112	—0,0072
1,000	1,1192	—0,1517	0,0466	—0,0217	0,0124	—0,0080
1,500	1,1537	—0,2013	0,0667	—0,0318	0,0184	—0,0119
2,000	1,1784	—0,2367	0,0848	—0,0414	0,0241	—0,0157
3,000	1,2102	—0,2881	0,1154	—0,0589	0,0351	—0,0231
4,000	1,2287	—0,3215	0,1396	—0,0750	0,0451	—0,0300
5,000	1,2403	—0,3442	0,1588	—0,0876	0,0543	—0,0366
6,000	1,2478	—0,3604	0,1740	—0,0991	0,0626	—0,0427
7,000	1,2532	—0,3722	0,1861	—0,1089	0,0701	—0,0483
8,000	1,2569	—0,3812	0,1959	—0,1174	0,0768	—0,0535
9,000	1,2598	—0,3880	0,2039	—0,1246	0,0828	—0,0583
10,000	1,2612	—0,3934	0,2104	—0,1309	0,0881	—0,0676
15,000	1,2677	—0,4084	0,2320	—0,1514	0,1072	—0,0795
20,000	1,2699	—0,4147	0,2394	—0,1621	0,1182	—0,0901
30,000	1,2717	—0,4198	0,2472	—0,1718	0,1291	—0,1015
40,000	1,2723	—0,4217	0,2502	—0,1759	0,1340	—0,1069
50,000	1,2727	—0,4227	0,2517	—0,1779	0,1365	—0,1098
60,000	1,2728	—0,4232	0,2526	—0,1791	0,1379	—0,1115
80,000	1,2730	—0,4237	0,2535	—0,1803	0,1394	—0,1132
100,000	1,2731	—0,4239	0,2539	—0,1808	0,1405	—0,1141
∞	1,2732	—0,4244	0,2546	—0,1819	0,1415	—0,1157

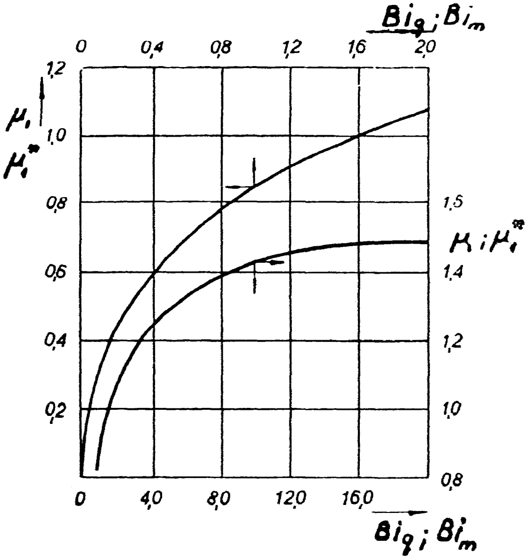


Fig. 25. First roots of characteristic equations $\operatorname{ctg}\mu = \frac{1}{\mathrm{Bi}_q}\mu$; and $\operatorname{ctg}\mu^* = \frac{1}{\mathrm{Bi}_m}\mu^*$ as a function of the Biot number for a plate.

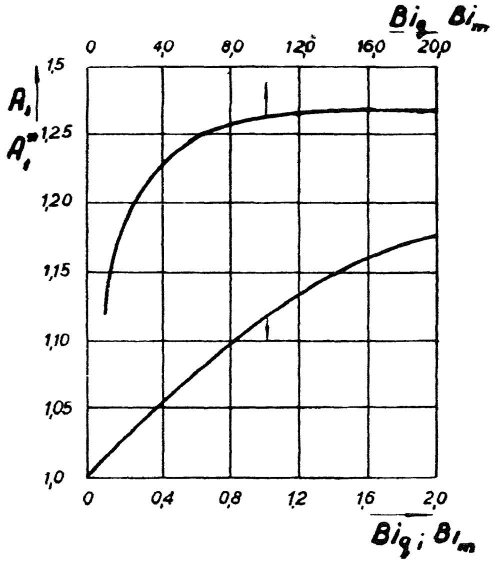


Fig. 26. A_1, and A_1^* as a function of the Biot number for a plate.

The solution for heat and mass transfer in a sphere at uniform initial conditions may be described as follows:

$$T=\frac{t\,(r,\,\tau)-t_0}{t_c-t_0}=1-$$

$$-\sum_{n=1}^{\infty}A_n\left(1+\frac{\varphi_n}{\psi_n}\right)\frac{\sin\mu_n\dfrac{r}{R}}{\mu_n\dfrac{r}{R}}\exp\left(-\mu_n^2\,\mathrm{Fo}\right)-$$

$$-\sum_{n=1}^{\infty}A_n^*\,\mathrm{Ko}\left(\frac{\varepsilon\,\mathrm{Lu}}{1-\mathrm{Lu}}\;\frac{\sin\mu_n^*\dfrac{r}{R}}{\mu_n^*\dfrac{r}{R}}+\right.$$

$$\left.+\frac{\varphi_n^*}{\psi_n^*}\;\frac{\sin\sqrt{\mathrm{Lu}}\,\mu_n^*\dfrac{r}{R}}{\mu_n^*\dfrac{r}{R}}\right)\exp\left(-\mu_n^{*\,2}\mathrm{Fo}^*\right),$$

$$\Theta=\frac{\theta_0-\theta\,(r,\,\tau)}{\theta_0-\theta_p}=1-$$

$$-\sum_{n=1}^{\infty}A_n^*\,\frac{\sin\mu_n^*\dfrac{r}{R}}{\mu_n^*\dfrac{r}{R}}\exp\left(-\mu_n^{*\,2}\,\mathrm{Fo}^*\right), \tag{5.4}$$

where

$$\varphi_n=\mathrm{KoLu}\,\frac{\mathrm{Bi}_m}{\mathrm{Bi}_q}\left\{\left[1-\varepsilon+\frac{\varepsilon}{\mathrm{Lu}-1}\left(\mathrm{Bi}_q-1\right)\right]\sin\frac{\mu_n}{\sqrt{\mathrm{Lu}}}-\right.$$

$$\left.-\left[1-\varepsilon-\frac{\varepsilon}{\mathrm{Lu}-1}\right]\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\cos\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right\},$$

$$\varphi_n^*=\mathrm{Lu}\left\{\left[1-\varepsilon+\frac{\varepsilon}{\mathrm{Lu}-1}\left(\mathrm{Bi}_q-1\right)\right]\sin\mu_n^*-\right.$$

$$\left.-\left[1-\varepsilon-\frac{\varepsilon}{\mathrm{Lu}-1}\right]\mu_n^*\cos\mu_n^*\right\},$$

$$\psi_n=\left(\mathrm{Bi}_m-1\right)\sin\frac{\mu_n}{\sqrt{\mathrm{Lu}}}+\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\cos\frac{\mu_n}{\sqrt{\mathrm{Lu}}},$$

$$\psi_n^*=\left(\mathrm{Bi}_q-1\right)\sin\sqrt{\mathrm{Lu}}\,\mu_n^*+\sqrt{\mathrm{Lu}}\,\mu_n^*\cos\sqrt{\mathrm{Lu}}\,\mu_n^*,$$

μ_n and μ_n^* are determined from the characteristic equations:

$$\operatorname{tg}\mu = -\frac{1}{\mathrm{Bi}_q - 1}\mu \text{ and } \operatorname{tg}\mu^* = -\frac{1}{\mathrm{Bi}_m - 1}\mu^*, \tag{5.5}$$

and the coefficients A_n and A_n^* are determined by

$$A_n = \frac{2(\sin\mu_n - \mu_n\cos\mu_n)}{\mu_n - \sin\mu_n\cos\mu_n},$$
$$A_n^* = \frac{2(\sin\mu_n^* - \mu_n^*\cos\mu_n^*)}{\mu_n^* - \sin\mu_n^*\cos\mu_n^*} \tag{5.6}$$

The values of the characteristic roots are given in Table 25 and Figure 27, and the coefficients A_n and A_n^* are given in Table 26 and Figure 28.

The following is the solution for heat and mass transfer in an infinite cylinder with uniform initial conditions:

$$T = \frac{t(r,\tau) - t_0}{t_c - t_0} = 1 - \sum_{n=1}^{\infty} A_n\left(1 + \frac{\varphi_n}{\psi_n}\right)J_0\left(\mu_n\frac{r}{R}\right)\times$$
$$\times\exp(-\mu_n^2\,\mathrm{Fo}) - \sum_{n=1}^{\infty} A_n^*\,\mathrm{Ko}\left[\frac{\varepsilon\,\mathrm{Lu}}{1-\mathrm{Lu}}J_0\left(\mu_n^*\frac{r}{R}\right) + \right.$$
$$\left. + \frac{\varphi_n^*}{\psi_n^*}J_0\left(\mu_n^*\sqrt{\mathrm{Lu}}\,\frac{r}{R}\right)\right]\exp(-\mu_n^{*2}\mathrm{Fo}^*),$$

$$\Theta = \frac{\theta_0 - \theta(r,\tau)}{\theta_0 - \theta_p} = 1 - \sum_{n=1}^{\infty} A_n^* J_0\left(\mu_n^*\frac{r}{R}\right)\exp(-\mu_n^{*2}\mathrm{Fo}^*), \tag{5.7}$$

where

$$\varphi_n = \mathrm{Ko}\left[\frac{\mathrm{Bi}_m}{\mathrm{Bi}_q}\left(1 - \varepsilon - \frac{\varepsilon}{\mathrm{Lu}-1}\right)\sqrt{\mathrm{Lu}}\,\mu_n J_1\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right) + \right.$$
$$\left. + \frac{\varepsilon\,\mathrm{Lu}\mathrm{Bi}_m}{\mathrm{Lu}-1}J_0\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right)\right],$$

$$\varphi_n^* = \mathrm{Lu}\left[\left(1 - \varepsilon - \frac{\varepsilon}{\mathrm{Lu}-1}\right)\mu_n^* J_1(\mu_n^*) + \frac{\varepsilon\,\mathrm{Bi}_q}{\mathrm{Lu}-1}J_0(\mu_n^*)\right],$$

$$\psi_n = \mathrm{Bi}_m J_0\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right) - \frac{\mu_n}{\sqrt{\mathrm{Lu}}}J_1\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right),$$

$$\psi_n^* = \mathrm{Bi}_q J_0(\sqrt{\mathrm{Lu}}\,\mu_n^*) - \sqrt{\mathrm{Lu}}\,\mu_n^* J_1(\sqrt{\mathrm{Lu}}\,\mu_n^*),$$

μ_n and μ_n^* are roots of the characteristic equations

$$\frac{J_0(\mu)}{J_1(\mu)} = \frac{\mu}{\mathrm{Bi}_q} \quad \text{and} \quad \frac{J_0(\mu^*)}{J_1(\mu^*)} = \frac{\mu^*}{\mathrm{Bi}_m}, \tag{5.8}$$

coefficients A_n and A_n^* are determined by the following:

$$A_n = \frac{2J_1(\mu_n)}{\mu_n\,[J_0^2(\mu_n) + J_1^2(\mu_n)]}, \quad A_n^* = \frac{2J_1(\mu_n^*)}{\mu_n^*\,[J_0^2(\mu_n^*) + J_1^2(\mu_n^*)]}. \tag{5.9}$$

Here, J_0 and J_1 are zero and first-order Bessel functions of the first kind.

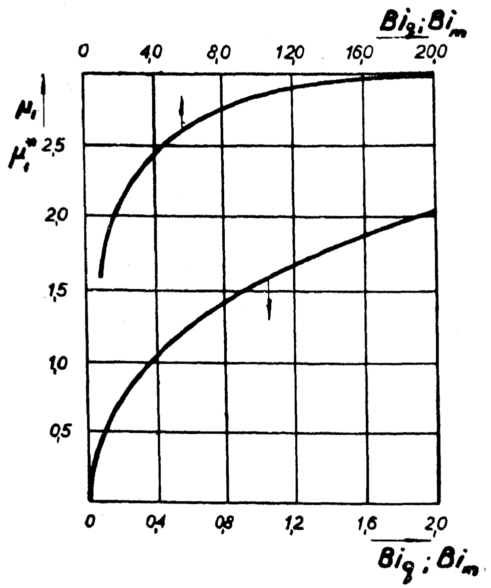


Fig. 27. First roots of characteristic equations $\mathrm{tg}\,\mu = -\frac{1}{\mathrm{Bi}_q - 1}\mu$ and $\mathrm{tg}\,\mu^* = -\frac{1}{\mathrm{Bi}_m}\mu^*$ as functions of the Biot number for a sphere.

The roots of characteristic equations (5.8) are given in Table 27 and the values of coefficients A_n and A_n^* are cited in Table 28. Graphical representations of the above values are given in Figures 29 and 30, respectively.

Table 25. Roots of the characteristic equations

$$\mathrm{tg}\,\mu = -\frac{1}{\mathrm{Bi}_q - 1}\,\mu \text{ and } \mathrm{tg}\,\mu^* = -\frac{1}{\mathrm{Bi}_m - 1}\,\mu^*$$

Bi_q Bi_m	μ_1 μ_1^*	μ_2 μ_2^*	μ_3 μ_3^*	μ_4 μ_4^*	μ_5 μ_5^*	μ_6 μ_6^*
0,000	0,0000	4,4934	7,7253	10,9041	14,0662	17,2208
0,005	0,1224	4,4945	7,7259	10,9046	14,0666	17,2210
0,010	0,1730	4,4956	7,7265	10,9050	14,0669	17,2213
0,020	0,2445	4,4979	7,7278	10,9060	14,0676	17,2219
0,030	0,2991	4,5001	7,7291	10,9069	14,0683	17,2225
0,040	0,3450	4,5023	7,7304	10,9078	14,0690	17,2231
0,050	0,3854	4,5045	7,7317	10,9087	14,0697	17,2237
0,060	0,4217	4,5068	7,7330	10,9096	14,0705	17,2242
0,070	0,4551	4,5090	7,7343	10,9105	14,0712	17,2248
0,080	0,4860	4,5112	7,7356	10,9115	14,0719	17,2254
0,090	0,5150	4,5134	7,7369	10,9124	14,0726	17,2260
0,100	0,5423	4,5157	7,7382	10,9133	14,0733	17,2266
0,150	0,6609	4,5268	7,7447	10,9179	14,0769	17,2295
0,200	0,7593	4,5379	7,7511	10,9225	14,0804	17,2324
0,300	0,9208	4,5601	7,7641	10,9316	14,0875	17,2382
0,400	1,0528	4,5822	7,7770	10,9408	14,0946	17,2440
0,500	1,1656	4,6042	7,7899	10,9499	14,1017	17,2498
0,600	1,2644	4,6261	7,8028	10,9591	14,1088	17,2556
0,700	1,3525	4,6479	7,8156	10,9682	14,1159	17,2614
0,800	1,4320	4,6696	7,8284	10,9774	14,1230	17,2672
0,900	1,5044	4,6911	7,8412	10,9865	14,1301	17,2730
1,000	1,5708	4,7124	7,8540	10,9956	14,1372	17,2788
1,100	1,6320	4,7335	7,8667	11,0047	14,1443	17,2845
1,200	1,6887	4,7544	7,8794	11,0137	14,1513	17,2903
1,300	1,7414	4,7751	7,8920	11,0228	14,1584	17,2961
1,400	1,7906	4,7956	7,9046	11,0318	14,1654	17,3019
1,500	1,8366	4,8158	7,9171	11,0409	14,1724	17.3076
1,600	1,8798	4,8358	7,9295	11,0498	14,1795	17,3134
1,700	1,9203	4,8556	7,9419	11,0588	14,1865	17,3192
1,800	1,9586	4,8751	7,9542	11,0677	14,1935	17,3249
1,900	1,9947	4,8943	7,9665	11,0767	14,2005	17,3306
2,000	2,0288	4,9132	7,9787	11,0856	14,2075	17,3364
2,500	2,1746	5,0037	8,0385	11,1296	14,2421	11,3649
3,000	2,2889	5,0870	8,0962	11,1727	14,2764	17,3932
4,000	2,4557	5,2329	8,2045	11,2560	14,3434	17,4490
5,000	2,5704	5,3540	8,3029	11,3349	14,4080	17,5034
6,000	2,6537	5,4544	8,3914	11,4086	14,4699	17,5562
7,000	2,7165	5,5378	8,4703	11,4773	14,5288	17,6072
8,000	2,7654	5,6078	8,5406	11,5408	14,5847	17,6567
9,000	2,8044	5,6669	8,6031	11,5994	14,6374	17,7032
10,000	2,8363	5,7172	8,6587	11,6532	14,6870	17,7481
11,000	2,8628	5,7606	8,7083	11,7027	14,7335	17,7908
16,000	2,9476	5,9080	8,8898	11,8959	14,9251	17,9742
21,000	2,9930	5,9921	9,0019	12,0250	15,0625	18,1136
31,000	3,0406	6,0831	9,1294	12,1807	15,2380	18,3018
41,000	3,0651	6,1311	9,1987	12,2688	15,3417	18,4180
51,000	3,0801	6,1606	9,2420	12,3247	15,4090	18,4953
61,000	3,0901	6,1805	9,2715	12,3632	15,4559	18,5497
81,000	3,1028	6.2058	9,3089	12.4124	15,5164	18,6209
101,000	3,1105	6,2211	9,3317	10,4426	15,5537	18,6650
∞	3,1416	6,2832	9,4248	12,5664	15,7080	18,8496

Table 26. Coefficients A_n and A_n^* for a sphere.

Bi_q Bi_m	A_1 A_1^*	A_2 A_2^*	A_3 A_3^*	A_4 A_4^*	A_5 A_5^*	A_6 A_6^*
0,000	1,0000	—0,0000	0,0000	—0,0000	0,0000	—0,0000
0,005	1,0025	—0,0023	0,0013	—0,0009	0,0007	—0,0006
0,010	1,0035	—0,0046	0,0026	—0,0018	0,0014	—0,0012
0,020	1,0055	—0,0091	0,0052	—0,0037	0,0029	—0,0023
0,030	1,0089	—0,0137	0,0078	—0,0055	0,0043	—0,0035
0,040	1,0121	—0,0182	0,0104	—0,0074	0,0057	—0,0047
0,050	1,0147	—0,0227	0.0130	—0,0092	0,0071	—0,0058
0,060	1,0181	—0,0273	0,0156	—0,0110	0,0085	—0,0070
0,070	1,0206	—0,0318	0,0183	—0,0129	0,0100	—0,0081
0,080	1,0239	—0,0363	0,0209	—0,0147	0,0114	—0,0093
0,090	1,0266	—0,0409	0,0235	—0,0166	0,0128	—0,0105
0,100	1,0297	—0,0454	0,0260	—0,0184	0,0142	—0,0116
0,150	1,0443	—0,0679	0,0390	—0,0276	0,0214	—0,0174
0,200	1,0592	—0,0894	0,0520	—0,0368	0,0285	—0,0232
0,300	1,0880	—0,1345	0,0779	—0,0551	0,0427	—0,0349
0,400	1,1164	—0,1781	0,1036	—0,0734	0,0569	—0,0465
0,500	1,1440	—0,2216	0,1292	—0,0916	0,0710	—0,0580
0,600	1,1713	—0,2633	0,1546	—0,1098	0,0852	—0,0696
0,700	1,1978	—0.3048	0,1799	—0,1270	0,0998	—0,0812
0,800	1,2237	—0,3455	0,2050	—0,1460	0,1134	—0,0927
0,900	1,2488	—0,3854	0,2299	—0,1640	0,1275	—0,1042
1,000	1,2732	—0,4244	0,2546	—0,1819	0,1415	—0,1157
1,100	1,2970	—0,4626	0,2792	—0,1997	0,1555	—0,1272
1,200	1,3200	—0,4999	0,3035	—0,2175	0,1694	—0,1387
1,300	1,3424	—0,5364	0,3276	—0,2352	0,1833	—0,1501
1,400	1,3640	—0,5720	0,3515	—0,2528	0,1972	—0,1616
1,500	1,3848	—0,6067	0,3752	—0,2703	0,2110	—0,1730
1,600	1,4051	—0,6405	0,3986	—0,2878	0,2248	—0,1843
1,700	1,4247	—0,6735	0,4218	—0,3051	0,2385	—0,1957
1,800	1,4436	—0,7063	0,4447	—0,3228	0,2522	—0,2078
1,900	1,4618	—0,7368	0,4674	—0,3395	0,2659	—0,2183
2,000	1,4793	—0,7673	0,4899	—0,3565	0,2795	—0,2296
2,500	1,5579	—0,9073	0,5980	—0,4365	0,3449	—0,2855
3,000	1,6223	—1,0288	0,6993	—0,5205	0,4122	—0,3405
4,000	1,7201	—1,2253	0,8811	—0,6719	0,5384	—0,4476
5,000	1,7870	—1,3733	1,0363	—0,8095	0,6570	—0,5501
6,000	1,8338	—1,4860	1,1673	—0,9333	0,7702	—0,6428
7,000	1,8673	—1,5731	1,2776	—1,0437	0,8695	—0,7398
8,000	1,8920	—1,6409	1,3703	—1,1415	0,9633	—0,8264
9,000	1,9106	—1,6949	1,4482	—1,2280	1,0489	—0,9073
10,000	1,9249	—1,7381	1,5141	—1,3042	1,1269	—0,9027
11,000	1,9364	—1,7732	1,5698	—1,3713	1,1977	—1,0527
16,000	1,9663	—1.8766	1,7489	—1,6058	1,4633	—1,3305
21.000	1,9801	—1,9235	1,8385	—1,7360	1,6256	—1,5149
31,000	1,9905	—1,9626	1,9186	—1,8616	1,7950	—1,7225
41,000	1,9948	—1,9780	1,9515	—1,9161	1,8732	—1,8263
51,000	1,9964	—1,9856	1,9680	—1,9441	1,9145	—1,8802
61,000	1,9974	—1,9901	1,9773	—1,9601	1,9387	—1,9135
81,000	1,9985	—1,9942	1,9869	—1,9769	1,9644	—1,9482
101,000	1,9993	—1,9962	1,9915	—1,9850	1,9767	—1,9667
∞	2,0000	—2,0000	2,0000	—2,0000	2,0000	—2,0000

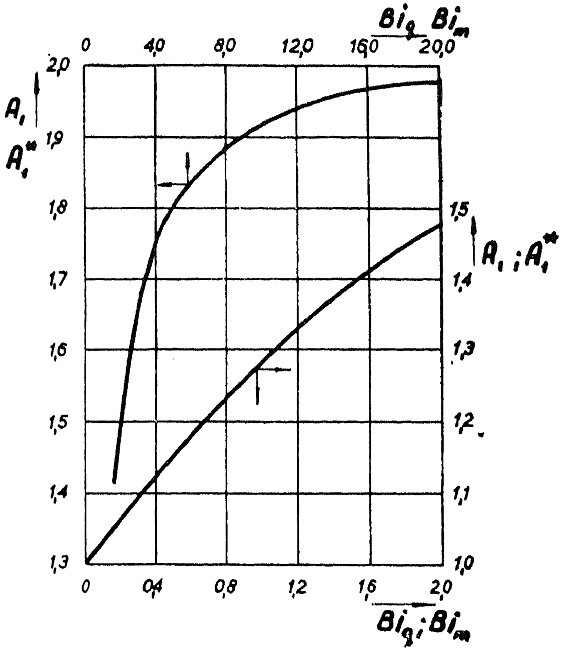


Fig. 28. Coefficients A_1, and A_1^* as functions of Biot numbers for a sphere.

b) Absence of Thermal Mass Transfer and of Phase Transition ($\delta = 0$, $\varepsilon = 0$).

For a plate,

$$T = \frac{t(x,\tau) - t_0}{t_c - t_0} = 1 -$$

$$-\sum_{n=1}^{\infty} A_n \left[1 + \frac{\mathrm{Ko}\sqrt{\mathrm{Lu}}\,\dfrac{\mathrm{Bi}_m}{\mathrm{Bi}_q}\,\mu_n \sin\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}}{\mathrm{Bi}_m \cos\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}} - \dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}\sin\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}} \right] \times$$

$$\times \cos \mu_n \frac{x}{R} \exp(-\mu_n^2 \mathrm{Fo}) -$$

$$-\sum_{n=1}^{\infty} A_n^* \frac{\mathrm{KoLu}\,\mu_n^* \sin \mu_n^* \cos\left(\sqrt{\mathrm{Lu}}\,\mu_n^* \dfrac{x}{R}\right)}{\mathrm{Bi}_q \cos\sqrt{\mathrm{Lu}}\,\mu_n^* - \sqrt{\mathrm{Lu}}\,\mu_n^* \sin\sqrt{\mathrm{Lu}}\,\mu_n^*} \exp(-\mu_n^{*2}\mathrm{Fo}^*).$$

Table 27. Roots of characteristic equations

$$\frac{J_0(\mu)}{J_1(\mu)} = \frac{1}{\mathrm{Bi}_q}\mu \text{ and } \frac{J_0(\mu^*)}{J_1(\mu^*)} = -\frac{1}{\mathrm{Bi}_m}\mu^*.$$

Bi_q	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
Bi_m	μ_1^*	μ_2^*	μ_3^*	μ_4^*	μ_5^*	μ_6^*
0,00	0,0000	3,8317	7,0156	10,1735	13,3237	16,4706
0,01	0,1412	3,8343	7,0170	10,1745	13,3244	16,4712
0,02	0,1995	3,8369	7,0184	10,1754	13,3252	16,4718
0,04	0,2814	3,8421	7,0213	10,1774	13,3267	16,4731
0,06	0,3438	3,8473	7,0241	10,1794	13,3282	16,4743
0,08	0,3960	3,8525	7,0270	10,1813	13,3297	16,4755
0,10	0,4417	3,8577	7,0298	10,1833	13,3312	16,4767
0,15	0 5376	3,8706	7,0369	10,1882	13,3349	16,4797
0,20	0,6170	3,8835	7,0440	10,1931	13,3387	16,4828
0,30	0,7465	3,9091	7,0582	10,2029	18,3462	16,4888
0,40	0,8516	3,9344	7,0723	10,2127	13,3537	16,4949
0,50	0,9408	3,9594	7,0864	10,2225	13,3611	16,5010
0,60	1,0184	3,9841	7,1004	10,2322	13,3686	16,5070
0,70	1,0873	4,0085	7,1143	10,2419	13,3761	16,5131
0,80	1,1490	4,0325	7,1282	10,2519	13,3835	16,5191
0,90	1,2048	4,0562	7,1421	10,2613	13,3910	16,5251
1,00	1,2558	4,0795	7,1558	10,2710	13,3984	16,5312
1,50	1,4569	4,1902	7,2233	10,3188	13,4353	16,5612
2,00	1,5994	4,2910	7,2884	10,3658	13,4719	16,5910
3,00	1,7887	4,4634	7,4103	10,4566	13,5434	16,6499
4,00	1,9081	4,6018	7,5201	10,5423	13,6125	16,7073
5,00	1,9898	4,7131	7,6177	10,6223	13,6786	16,7630
6,00	2,0490	4,8033	7,7039	10,6964	13,7414	16,8168
7,00	2,0937	4,8772	7,7797	10,7646	13,8008	16,8684
8,00	2,1286	4,9384	7,8464	10,8271	13,8566	16,9179
9,00	2,1566	4,9897	7,9051	10,8842	13,9090	16,9650
10,00	2,1795	5,0332	7,9569	10,9363	13,9580	17,0099
15,00	2,2509	5,1773	8,1422	11,1367	14,1576	17,2008
20,00	2,2880	5,2568	8,2534	11,2677	14,2983	17,3442
30,00	2,3261	5,3410	8,3771	11,4221	14,4748	17,5348
40,00	2,3455	5,3846	8,4432	11,5081	14,5774	17,6508
50,00	2,3572	5,4112	8,4840	11,5621	14,6433	17,7272
60,00	2,3651	5,4291	8,5116	11,5990	14,6889	17,7807
80,00	2,3750	5,4516	8,5466	11,6461	14,7475	17,8502
100,00	2,3809	5,4652	8,5678	11,6747	14,7834	17,8931
∞	2,4048	5,5201	8,6537	11,7915	14,9309	18,0711

Table 28. Values of coefficients A_n and A_n^* for an infinite cylinder.

Bi_q	$+A_1$	$-A_2$	$+A_3$	$-A_4$	$+A_5$	$-A_6$
Bi_m	$+A_1^*$	$-A_2^*$	$+A_3^*$	$-A_4^*$	$+A_5^*$	$-A_6^*$
0,00	1,0000	0,0000	0,0000	0,0000	0,0000	0,0000
0,01	1,0031	0,0034	0,0013	0,0008	0,0005	0,0004
0,02	1,0049	0,0067	0,0027	0,0015	0,0010	0,0007
0,04	1,0102	0,0135	0,0052	0,0031	0,0021	0,0015
0,06	1,0150	0,0201	0,0081	0,0046	0,0031	0,0023
0,08	1,0199	0,0268	0,0110	0,0062	0,0041	0,0030
0,10	1,0245	0,0333	0,0135	0,0077	0,0051	0,0037
0,15	1,0366	0,0497	0,0202	0,0116	0,0077	0,0056
0,20	1,0482	0,0658	0,0269	0,0154	0,0103	0,0075
0,30	1,0711	0,0972	0,0401	0,0231	0,0155	0,0112
0,40	1,0931	0,1277	0,0582	0,0307	0,0205	0,0150
0,50	1,1142	0,1571	0,0662	0,0383	0,0256	0,0187
0,60	1,1345	0,1857	0,0790	0,0458	0,0307	0,0224
0,70	1,1539	0,2132	0,0917	0,0533	0,0358	0,0261
0,80	1,1724	0,2398	0,1043	0,0608	0,0408	0,0298
0,90	1,1902	0,2654	0,1167	0,0682	0,0459	0,0335
1,00	1,2071	0,2901	0,1289	0,0756	0,0509	0,0372
1,50	1,2807	0,4008	0,1877	0,1117	0,0756	0,0554
2,00	1,3377	0,4923	0,2422	0,1404	0,0998	0,0732
3,00	1,4192	0,6309	0,3384	0,2114	0,1463	0,1084
4,00	1,4698	0,7278	0,4184	0,2699	0,1898	0,1420
5,00	1,5029	0,7973	0,4842	0,3220	0,2301	0,1735
6,00	1,5253	0,8484	0,5382	0,3679	0,2672	0,2038
7,50	1,5409	0,8869	0,5825	0,4080	0,3010	0,2317
8,00	1,5523	0,9225	0,6189	0,4430	0,3316	0,2579
9,00	1,5611	0,9393	0,6491	0,4735	0,3593	0,2826
10,00	1,5677	0,9575	0,6784	6,5000	0,3843	0,3042
15,00	1,5853	1,0091	0,7519	0,5901	0,4760	0,3913
20,00	1,5918	1,0309	0,7889	0,6382	0,5303	0,4461
30,00	1,5964	1,0488	0,8195	0,6827	0,5853	0,5062
40,00	1,5988	1,0550	0,8335	0,7018	0,6133	0,5390
50,00	1,5995	1,0587	0,8396	0,7112	0,6227	0,5544
60,00	1,6009	1,0589	0,8428	0,7165	0,6301	0,5642
80,00	1,6012	1,0599	0,8463	0,7212	0,6398	0,5770
100,00	1,6014	1,0631	0,8505	0,7245	0,6415	0,5850
∞	1,6021	1,0648	0,8558	0,7296	0,6485	0,5896

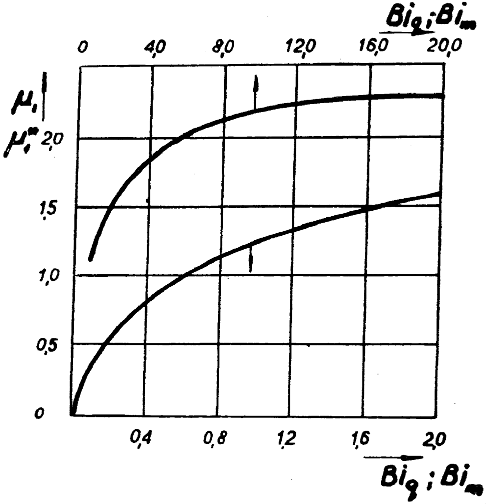


Fig. 29. First roots of characteristic equations $\frac{J_0(\mu)}{J_1(\mu)} = \frac{\mu}{\mathrm{Bi}_q}$ and $\frac{J_0(\mu^*)}{J_1(\mu^*)} = \frac{\mu^*}{\mathrm{Bi}_m}$ as a function of Biot numbers for an infinite cylinder.

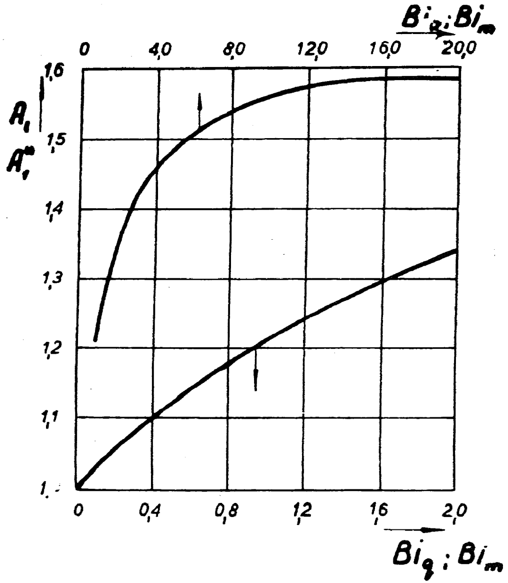


Fig. 30. A_1, and A_1^* as functions of Biot numbers for an infinite cylinder.

For a sphere,

$$T = \frac{t\,(r,\tau) - t_0}{t_c - t_0} = 1 -$$

$$- \sum_{n=1}^{\infty} A_n \left[1 + \frac{\mathrm{KoLu}\,\dfrac{\mathrm{Bi}_m}{\mathrm{Bi}_q}\left(\sin\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}} - \dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}\cos\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}\right)}{(\mathrm{Bi}_m - 1)\sin\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}} + \dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}\cos\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}} \right] \times$$

$$\times \frac{\sin \mu_n \dfrac{r}{R}}{\mu_n \dfrac{r}{R}} \exp(-\mu_n^2\,\mathrm{Fo}) -$$

$$- \sum_{n=1}^{\infty} A_n^* \frac{\mathrm{Ko\,Lu\,Bi}_m \sin \mu_n^*}{(\mathrm{Bi}_q - 1)\sin\sqrt{\mathrm{Lu}}\,\mu_n^* + \sqrt{\mathrm{Lu}}\,\mu_n^* \cos\sqrt{\mathrm{Lu}}\,\mu_n^*} \times$$

$$\times \frac{\sin\sqrt{\mathrm{Lu}}\,\mu_n^* \dfrac{r}{R}}{\mu_n^* \dfrac{r}{R}} \exp(-\mu_n^{*2}\mathrm{Fo}^*).$$

For a cylinder,

$$T = \frac{t\,(r,\tau) - t_0}{t_c - t_0} =$$

$$= 1 - \sum_{n=1}^{\infty} A_n \left[1 + \frac{\mathrm{Ko}\sqrt{\mathrm{Lu}}\,\dfrac{\mathrm{Bi}_m}{\mathrm{Bi}_q}\,\mu_n J_1\left(\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}\right)}{\mathrm{Bi}_m J_0\left(\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}\right) - \dfrac{\mu_n}{\sqrt{\mathrm{Lu}}} J_1\left(\dfrac{\mu_n}{\sqrt{\mathrm{Lu}}}\right)} \right] \times$$

$$\times J_0\left(\mu_n \frac{r}{R}\right) \exp\left(-\mu_n^2\,\mathrm{Fo}\right) -$$

$$- \sum_{n=1}^{\infty} A_n^* \frac{\mathrm{Ko\,Lu}\,\mu_n^* J_1(\mu_n^*)\,J_0\left(\mu_n^* \sqrt{\mathrm{Lu}}\dfrac{r}{R}\right)}{\mathrm{Bi}_q J_0(\sqrt{\mathrm{Lu}}\,\mu_n^*) - \sqrt{\mathrm{Lu}}\,\mu_n^* J_1(\sqrt{\mathrm{Lu}}\,\mu_n^*)} \exp(-\mu_n^{*2}\mathrm{Fo}^*).$$

The expressions for the fields of mass content are determined by equations (5.1), (5.4) and (5.7). The roots of characteristic equations are determined from equations (5.2), (5.5) and (5.8). The heat and mass transfer coefficients are determined by equations (5.3), (5.6) and (5.9).

In the absence of mass transfer ($\mathrm{Ko} = 0$) the solutions given above reduce to the usual classical solutions for temperature distributions in a plate, sphere, and cylinder.

c) Absence of Thermal Mass Transfer with Constant Rate of Mass Removal

$$(\delta = 0,\ q_m = \beta\,[\theta(R,\tau) - \theta_p] = \mathrm{const}).$$

This problem refers to constant-rate drying process. A solution of this problem given by F.M. Polonskaya can be considered the first step in solving the system of differential heat and mass transfer equations (I and II). The solution of the equation system for the distribution of mass-transfer potential in an infinite plate has the following form:

$$\frac{\theta_0 - \theta(x, \tau)}{\theta_0 - \theta_p} = \mathrm{Ki}_m \left[\mathrm{Fo}^* - \frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right) + \right.$$

$$\left. + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2 \pi^2} \cos n\pi \frac{x}{R} \exp(-n^2 \pi^2 \mathrm{Fo}^*) \right], \tag{5.10}$$

where $\mathrm{Ki}_m = \frac{q_m R}{\lambda_m (\theta_0 - \theta_p)}$ is the mass transfer Kirpichev number.

As the dimensionless time (Fo*) increases the term of (5.10) containing the infinite series diminishes rapidly. Thus, beginning with $\mathrm{Fo}^* = \mathrm{Fo\,Lu} = \frac{a_m \tau}{R^2} \geqslant 0{,}54$, the infinite series can be neglected because it is small in comparison with the first two terms of equation (5.10). That is, a quasistationary regime is established:

$$\frac{\theta_0 - \theta(x, \tau)}{\theta_0 - \theta_p} = \mathrm{Ki}_m \left[\mathrm{Fo}^* - \frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right)\right] \text{ for } \mathrm{Fo}^* > 0{,}54.$$

The solution for the temperature field is

$$\frac{t(x, \tau) - t_0}{t_м - t_0} = 1 - \frac{1}{2} \varepsilon \,\mathrm{Lu\,Ko\,Ki}_m \left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right) -$$

$$- \sum_{n=1}^{\infty} A_n \left(1 - \frac{1}{\mu_n^2} \varepsilon \,\mathrm{Lu\,Ko\,Ki}_m\right) \cos \mu_n \frac{x}{R} \exp(-\mu_n^2 \mathrm{Fo}). \tag{5.11}$$

Here, μ_n and A_n are determined by equations (5.1) and (5.3), and $t_м = t_c - \frac{(1-\varepsilon)\rho q_m}{\alpha}$. For values of $\mathrm{Fo} \geqslant 0.32$, the solution of equation (5.11) is simplified and the expression for the dimensionless temperature assumes the following form:

$$\frac{t(x, \tau) - t_0}{t_м - t_0} = 1 - \frac{1}{2} \varepsilon \,\mathrm{Lu\,Ko\,Ki}_m \left(1 - \frac{x^2}{R^2} + \frac{2}{\mathrm{Bi}_q}\right).$$

Consequently, the temperature field becomes stationary. The experiments of F.M. Polonskaya have verified the above temperature distribution in moist materials.

A more detailed analysis of the last relationship shows that the phase transformation number is given by

$$\varepsilon = \frac{2\lambda}{\rho q_m R}(t_{\text{п}} - t_{\text{ц}}).$$

This means that the phase transformation number can be determined from a known temperature drop and the physical properties of the material. Consequently, the flow of mass in the vapor state can also be determined. In accordance with the experiments of F. M. Polonskaya and P. V. Lebedev ε for gypsum plates amounts to 0.045; for clay: 0.75-1.0; for sand: 0.2-0.4; for cellulose: 0.09-0.2.

d) Generalized Boundary Conditions of the Third Kind in the Absence of Thermal Mass Transfer.

The previous problems were concerned with the analytical solutions of various cases of heat and mass transfer. In the solutions the rate of removal of mass from the material, q_m, was assumed to be either constant, $q_m = \text{const}$, or a function of the mass transfer potential $q_m = \beta[\theta(R,\tau) - \theta_p]$. The term which describes the rate of removal of mass into the surrounding medium (i.e. the boundary conditions), can assume various forms depending on the conditions of the process. The value of q_m can be a function of the coordinates and time. It is therefore very desirable to have one solution which would be applicable for various boundary conditions. For the case of $q_m = q_m(\tau)$, $\delta = 0$, such a solution for an infinite plate was obtained by A. V. Lykov and M. S. Smirnov. The boundary conditions for this problem are

$$-\lambda_q \frac{\partial t(R,\tau)}{\partial x} + \alpha\,[t_c - t(R,\tau)] - (1-\varepsilon)\,\rho\, q_m(\tau) = 0, \tag{5.12}$$

$$\lambda_m \frac{\partial \theta(R,\tau)}{\partial x} + q_m(\tau) = 0, \tag{5.13}$$

$$t(x,0) = t_0 = \text{const}, \quad \theta(x,0) = \theta_0 = \text{const}, \tag{5.14}$$

$$\frac{\partial \theta(0,\tau)}{\partial x} = \frac{\partial t(0,\tau)}{\partial x} = 0. \tag{5.15}$$

The solution of the system of equations (I) and (II), obtained by the method of integral transformation, with boundary conditions (5.12) to (5.15), has the following form:

$$\frac{\theta(x,\tau)}{\theta_0} = 1 - \int_0^\tau \frac{a_m}{R^2}\mathrm{Ki}_m(\eta)\,d\eta + \mathrm{Ki}_m(\tau)\frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right) +$$

$$+\sum_{n=1}^{\infty}(-1)^n \frac{2}{n^2\pi^2}\cos n\pi\frac{x}{R}\exp(-n^2\pi^2\,\mathrm{Fo}^*) \times$$

$$\times\left\{ \mathrm{Ki}_m(0) + \int_0^{\tau} \exp\left(n^2\pi^2 \frac{a_m \eta}{R^2} \right) \frac{\mathrm{Ki}_m(\eta)}{d\eta}\, d\eta \right\}. \qquad (5.16)$$

After a certain period of time, i.e. when $\mathrm{Fo}^* > \mathrm{Fo}^*_{\mathrm{I}}$, the infinite series can be neglected in formula (5.16) and only the first two terms remain.

Thus, the temperature field in an infinite plate may be described in the following manner:

$$\frac{t(x,\tau) - t_0}{t_c - t_0} =$$

$$= 1 - \mathrm{Ki}_m(\tau)\,\mathrm{Ko}\,\mathrm{Lu}\,\frac{t_c}{t_c - t_0}\left[0{,}5\varepsilon\left(1 - \frac{x^2}{R^2}\right) + \frac{1}{\mathrm{Bi}_q}\right]. \qquad (5.17)$$

In equations (5.16) and (5.17), $\mathrm{Ki}_m(\tau) = \frac{q_m(\tau)\,R}{\lambda_m \theta_0}$ is the mass transfer Kirpichev number, and $\mathrm{Ko} = \frac{\rho c_m \theta_0}{c_q t_c}$ is the Kossovich number.

We shall now find the average (integral) value for the transfer potentials:

$$\frac{\bar{\theta}(\tau)}{\theta_0} = 1 - \int_0^{\tau} \frac{a_m}{R^2}\,\mathrm{Ki}_m(\eta)\, d\eta,$$

$$\frac{\bar{t}(\tau) - t_0}{t_c - t_0} = 1 - \mathrm{Ki}_m(\tau)\,\mathrm{Ko}\,\mathrm{Lu}\,\frac{t_c}{t_c - t_0}\left[\frac{1}{3}\varepsilon + \frac{1}{\mathrm{Bi}_q}\right].$$

From the above relationship we get the following expression for the rate of change of the mass transfer potential (for example, in drying processes this corresponds to the rate of drying):

$$-\frac{d\bar{u}(\tau)}{d\tau} = \frac{u_0 a_m [t_c - \bar{t}(\tau)]}{R^2\,\mathrm{Ko}\,\mathrm{Lu}\,t_c\left(\frac{1}{3}\varepsilon + \mathrm{Bi}_q^{-1}\right)} =$$

$$= \frac{\lambda_q[\bar{t}_c - \bar{t}(\tau)]}{\rho \gamma_0 R^2 \left(\frac{1}{3}\varepsilon + \mathrm{Bi}_q^{-1}\right)}. \qquad (5.18)$$

Relationship (5.18) is of great practical importance because the rate of drying can be determined from the average temperature of the body and from heat transfer coefficients. In other words, this equation can be used to reduce the calculations of mass transfer rate to the calculation of heat transfer. The above relationships can also be used to obtain the formula for the mass transfer Kirpichev number:

$$\mathrm{Ki}_m(\tau) = \frac{\lambda_q \left[t_c - \bar{t}(\tau)\right]}{\rho\, a_m \gamma_0 u_0 \left(\frac{1}{3}\varepsilon + \mathrm{Bi}_q^{-1}\right)}.$$

It follows from the above equation that the rate of vaporization is a function of the phase transformation number and of the heat and mass transfer coefficients.

6. THE EFFECT OF DIMENSIONLESS PARAMETERS ON HEAT AND MASS TRANSFER PROCESSES

The solutions of the system of differential equations for heat and mass transfer depend on a large number of dimensionless parameters. However, not all of the dimensionless numbers affect a given process to the same degree. Analysis of the results of a large series of numerical calculations of equations (I) to (VI) shows the predominant influence of some of the dimensionless numbers (Bi_m, Pn) on mass transfer, and of others of them (Bi_q, Ko, ε, W) on heat transfer in the same processes.

The most essential influence on the processes of heat and mass transfer is exerted by the Lu number, which can also be called the coupling parameter for heat and mass transfer. Basically, the Lu number characterizes the inertial properties of the mass transfer potential field in comparison with the inertial properties of the temperature field. Figure 31 shows that at low values of Lu the temperature distribution leads the distribution of mass transfer potential; at $\mathrm{Lu} \approx 1.0$ the two roughly coincide; and at $\mathrm{Lu} > 1.0$ the distribution of mass-transfer potential begins to lead the temperature distribution. The nature of the comparisons is not completely symmetric about $\mathrm{Lu} = 1.0$, however. To illustrate this point, let us contrast two cases: $\mathrm{Lu} = 0.5$ and $\mathrm{Lu} = 2.0$, i.e., the thermal diffusivity is either twice as large or only half as large as the mass diffusivity.

Table 29 and Figure 32 show that the difference between average dimensionless potentials is ± 12%, and their rates of change differ by up to 200%. Complete equality of heat and mass transfer potentials is not even observed at $\mathrm{Lu} = 1.0$; this occurs only at $\mathrm{Lu} < 1{,}0$. With increasing Lu, and at equal values of dimensionless time, the average temperature decreases and the drying rate increases; all processes proceed more uniformly during the period in which the mass transfer potential is changing (Fig. 33).

Consequently, from an engineering point of view the heat and

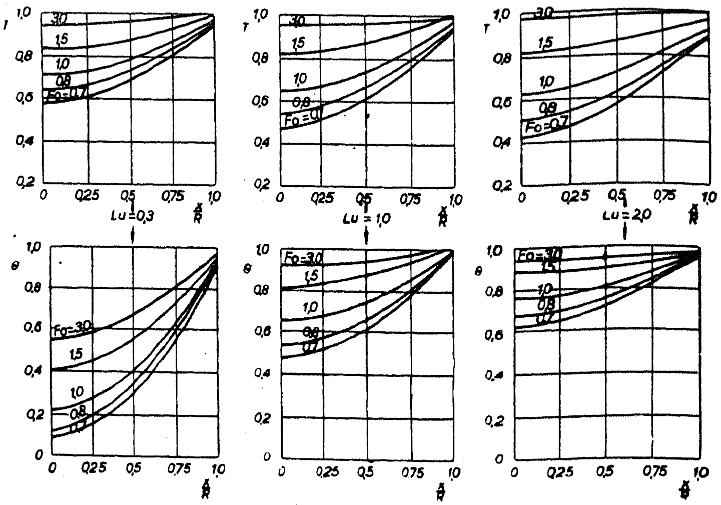


Fig. 31. Distribution of transfer potentials as a function of Lu.

Table 29

Fo	$\overline{T}:\overline{\Theta}$	$\overline{\Theta}:\overline{T}$	$\frac{d\overline{T}}{d\,\mathrm{Fo}}:\frac{d\overline{\Theta}}{d\mathrm{Fo}}$	$\frac{d\overline{\Theta}}{d\,\mathrm{Fo}}:\frac{d\overline{T}}{d\,\mathrm{Fo}}$	$\overline{T}:\overline{\Theta}$	$\frac{d\overline{T}}{d\mathrm{Fo}}:\frac{d\overline{\Theta}}{d\mathrm{Fo}}$
	Lu=0,5	Lu=2,0	Lu=0,5	Lu=2,0	Lu=1,0	
0,4	1,44	1,61			0,908	
0,6	1,41	1.35	1,23	0,63	0,946	1,10
0,8	1,35	1,22	0,22	0,60	0,965	1,09
1,0	1,29	1,15	0,79	0,59	0,976	1.08

mass transfer processes should be designed with large values of Lu. Large values of Lu will intensify the heating and drying process and decrease the possibility of thermal decomposition and other undesirable effects which occur at high temperatures.

The progress of the process is seriously affected by the dimensionless numbers describing surface and internal heat and mass

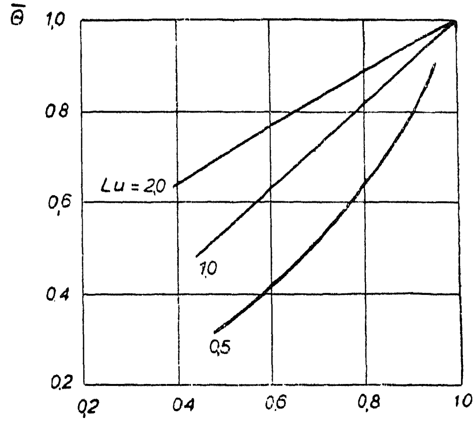


Fig. 32. Symmetry of $\bar{T}$ and $\bar{\Theta}$ with respect to Lu = 1.0.

transfer. The dimensionless heat and mass transfer numbers (Bi_q and Bi_m) characterize the rate of the external transfer in comparison with the rate of the internal transfer. At low values of the Biot number (see Table 30) the rate of heating and drying is insignificant, the temperature gradients and mass-potential gradients being small. With increasing Bi_q and Bi_m the transfer rates and simultaneously

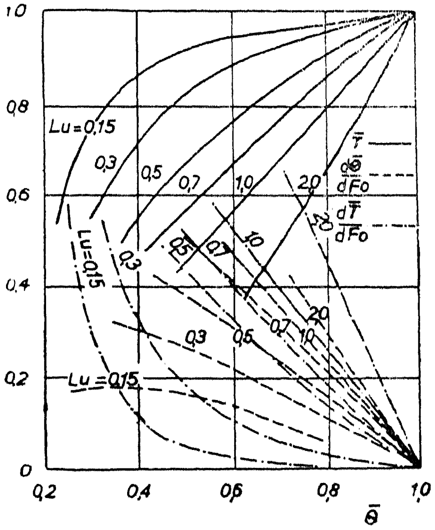


Fig. 33. Heat and mass transfer characteristics as a function of Lu.

Table 30. Dependence of heat and mass transfer characteristics on the Biot Number

Index	Fo	$\mathrm{Bi}_q = 0{,}5;\ \mathrm{Bi}_m = 0{,}5$			
		$\frac{x}{R} = 0$	$\frac{x}{R} = 0{,}25$	$\frac{x}{R} = 0{,}50$	$\frac{x}{R} = 0{,}75$
$T = \frac{t - t_0}{t_c - t_0}$	0,7	0,123	0,133	0,162	0,211
	0,8	0,147	0,157	0,185	0,232
	1,0	0.192	0.201	0,228	0,272
	1,5	0,293	0,301	0,324	0,362
	2,0	0,377	0,383	0,404	0,437
	3,0	0,507	0,513	0,528	0,553
	5,0	0,673	0,676	0,686	0,702
	7,0	0.768	0,771	0,777	0,788
	10,0	0,852	0,853	0,857	0,864
$\Theta = \frac{\theta_0 - \theta}{\theta_0 - \theta_p}$	0,7	—0,006	$0{,}118 \cdot 10^{-1}$	$0{,}640 \cdot 10^{-1}$	0,150
	0,8	$0{,}59 \cdot 10^{-2}$	$0{,}240 \cdot 10^{-1}$	$0{,}753 \cdot 10^{-1}$	0,160
	1,0	$0{,}303 \cdot 10^{-1}$	$0{,}448 \cdot 10^{-1}$	$0{,}974 \cdot 10^{-1}$	0,179
	1,5	$0{,}889 \cdot 10^{-1}$	0,105	0,151	0,226
	2,0	0,144	0,159	0,201	0,271
	3,0	0,244	0,257	0,293	0,353
	5,0	0,411	0,420	0,447	0,492
	7,0	0,541	0,548	0,569	0,603
	10,0	0,684	0,689	0,703	0,726
$\frac{dT}{d\,\mathrm{Fo}}$	1,0	0,219	0,216	0,208	0,195
	1,5	0,183	0,181	0,174	0,163
	2,0	0,154	0,152	0,146	0,137
	3,0	0,110	0,109	0,105	$0{,}987 \cdot 10^{-1}$
	5,0	$0{,}611 \cdot 10^{-1}$	$0{,}604 \cdot 10^{-1}$	$0{,}584 \cdot 10^{-1}$	$0.550 \cdot 10^{-1}$
$\frac{d\Theta}{d\,\mathrm{Fo}}$	1,0	0,121	0,118	0,109	$0{,}955 \cdot 10^{-1}$
	1,5	0,113	0,111	0,103	$0{,}911 \cdot 10^{-1}$
	2.0	0,107	0,104	$0{,}975 \cdot 10^{-1}$	$0{,}866 \cdot 10^{-1}$
	3,0	$0{,}941 \cdot 10^{-1}$	$0{,}922 \cdot 10^{-1}$	$0{,}868 \cdot 10^{-1}$	$0{,}779 \cdot 10^{-1}$
	5,0	$0{,}734 \cdot 10^{-1}$	$0{,}721 \cdot 10^{-1}$	$0{,}683 \cdot 10^{-1}$	$0{,}621 \cdot 10^{-1}$

Table 30, continued.

	$\mathrm{Bi}_q = 10$; $\mathrm{Bi}_m = 10$				
$\frac{x}{R}=1{,}00$	$\frac{x}{R}=0$	$\frac{x}{R}=0{,}25$	$\frac{x}{R}=0{,}50$	$\frac{x}{R}=0{,}75$	$\frac{x}{R}=1{,}00$
0,276	0,567	0,594	0,670	0,785	0,924
0,295	0,619	0,643	0,709	0,810	0,932
0,332	0,699	0,717	0,769	0,848	0,944
0,413	0,814	0,825	0,857	0,904	0,963
0,481	0,872	0,880	0,901	0,934	0,973
0,588	0,931	0,935	0,947	0,964	0,985
0,723	0,978	0,979	0,983	0,988	0,995
0,803	0,993	0,993	0,994	0,996	0,998
0,873	0,999	0,999	0,999	0,999	1,000
0,265	0,714·10^{-1}	0,134	0,311	0,579	0,898
0,273	0,118	0,177	0,344	0,598	0,901
0,289	0,206	0,259	0,408	0,635	0,907
0,327	0,394	0,433	0,546	0,718	0,926
0,364	0,539	0,568	0,654	0,784	0,943
0,433	0,734	0,751	0,800	0,876	0,967
0,553	0,912	0,918	0,934	0,959	0,989
0,649	0,971	0,973	0,978	0,986	0,996
0,758	0,995	0,995	0,996	0,998	0,999
0,177	0,333	0,181	0,147	0,953·10^{-1}	0,515·10^{-1}
0,148	0,156	0,981·10^{-1}	0,801·10^{-1}	0,529·10^{-1}	0,272·10^{-1}
0,125	0,874·10^{-1}	0,619·10^{-1}	0,508·10^{-1}	0,339·10^{-1}	0,169·10^{-1}
0,901·10^{-1}	0,397·10^{-1}	0,313·10^{-1}	0,258·10^{-1}	0,174·10^{-1}	0,841·10^{-2}
0,505·10^{-1}	0,124·10^{-1}	0,101·10^{-1}	0,830·10^{-2}	0,561·10^{-2}	0,569·10^{-2}
0,777·10^{-1}	0,421	0,347	0,274	0,167	0,357·10^{-1}
0,751·10^{-1}	0,330	0,268	0,214	0,132	0,362.10^{-1}
0,723·10^{-1}	0,254	0,205	0,164	0,102	0,303·10^{-1}
0,661·10^{-1}	0,147	0,118	0,950·10^{-1}	0,592·10^{-1}	0,184·10^{-1}
0,539·10^{-1}	0,486·10^{-1}	0 391·10^{-1}	0,314·10^{-2}	0,196·10^{-1}	0,614·10^{-2}

the gradients of transfer potentials increase. Following stabilization of the processes inside the material (for a plate $\mathrm{Fo} \approx 0.2$; for a sphere $\mathrm{Fo} \approx 0.4$) the heat transfer Biot number begins to influence only the thermal characteristics of heat and mass transfer $\left(T, \frac{dT}{d\mathrm{Fo}}\right)$ and does not influence the mass transfer characteristics $\left(\Theta, \frac{d\Theta}{d\mathrm{Fo}}\right)$. Conversely, the mass transfer Biot number influences only the mass transfer potentials (see Figs. 34 and 35). For all calculated values of Bi_q the drying rate could be represented by a single curve (see Fig. 36).

Figure 37 shows that for an increasing $\overline{\Theta}$ it is necessary that Bi_m increase and Bi_q decrease. The rate of change of both dimensionless numbers is approximately the same. The experimental data for Bi_m obtained by P.B. Lebedev and A.F. Chizhskiy agree with

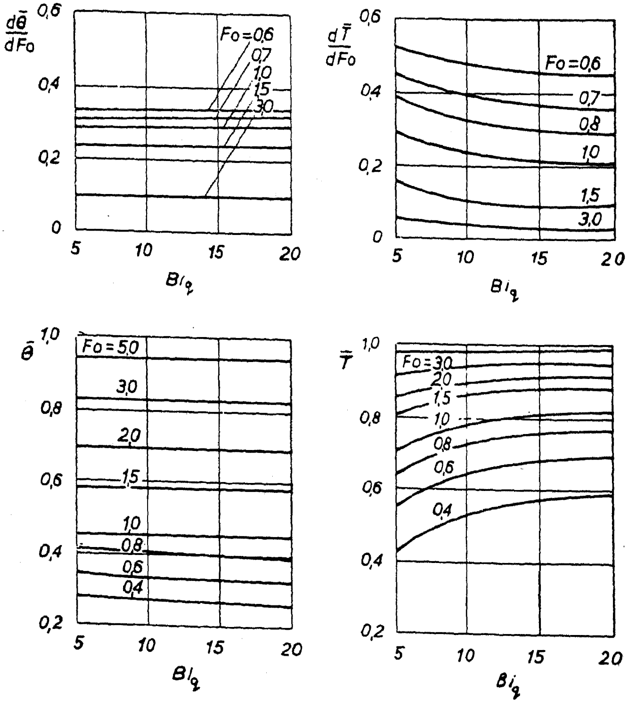


Fig. 34. Heat and mass transfer potentials as a function of Bi_q for an infinite plate.

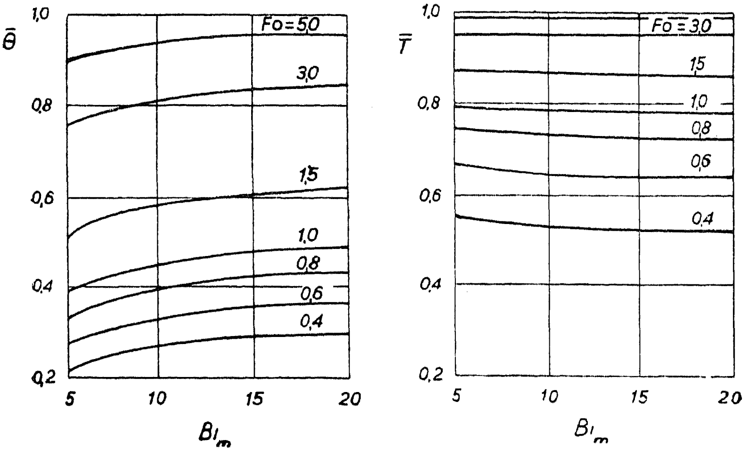


Fig. 35. Heat and mass transfer kinetics as a function of Bi_m for an infinite plate.

these results. The opposite situation obtains for increasing $\bar{T}$ it is necessary that Bi_m decrease and Bi_q increase. Under actual process conditions both the mass transfer and heat transfer Biot numbers are dependent on each other. Their relationship can be expressed mathematically by means of heat and mass balances. A change in the heat transfer Biot number requires a change in the mass transfer Biot number, although their numerical values are of the same order of magnitude.

With increase in both Biot numbers, the temperature increases considerably, the rate of heating increases and at lower values of $\bar{\Theta}$

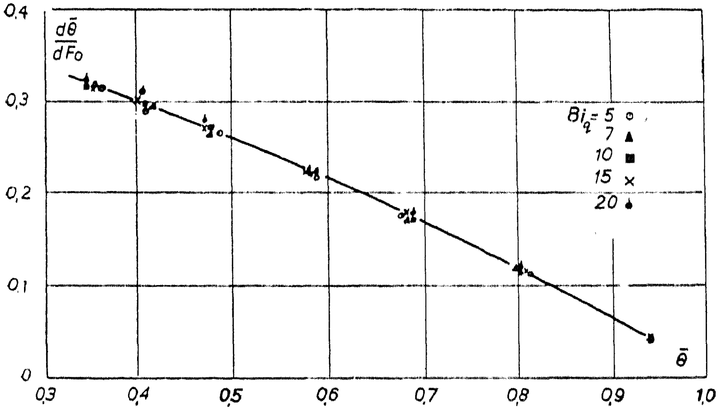


Fig. 36. Drying rate as a function of Bi_q.

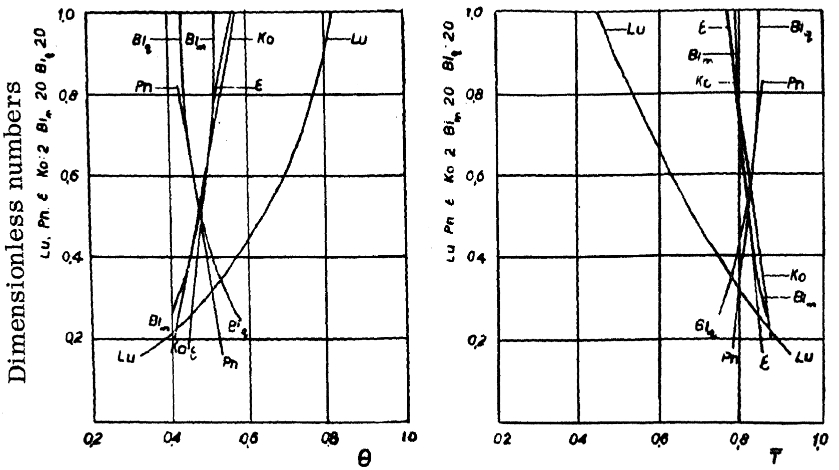
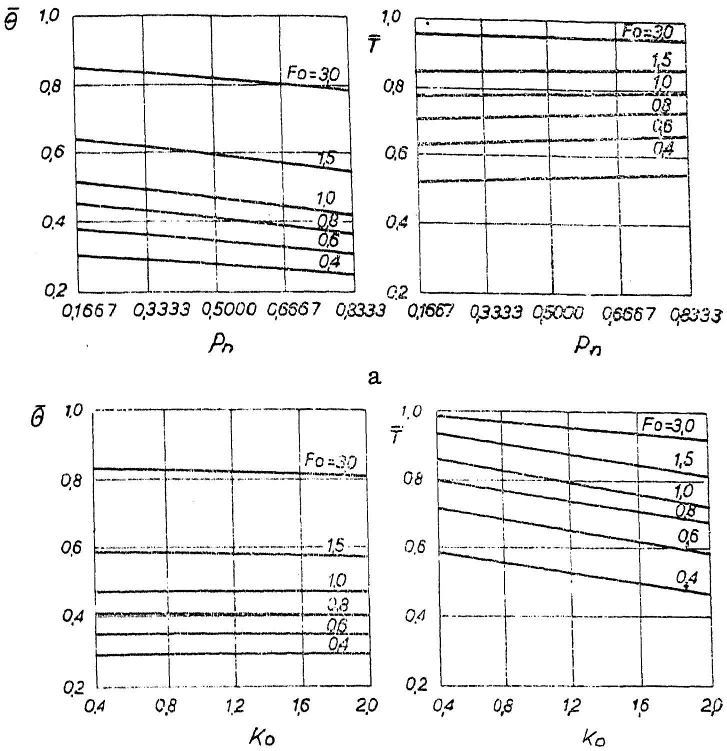


Fig. 37. Dimensionless heat and mass transfer numbers as a function of $\bar{T}$ and $\bar{\theta}$.

a stationary regime prevails. Drying proceeds at a faster rate and deeper layers of the material are affected.

Among the dimensionless numbers for heat and mass transfer we also have the Kossovich (Ko) number and the Posnov (Pn) number. These two dimensionless numbers resemble the Biot number, but they characterize only the internal heat and mass transfer without any definite connection with the process occurring in the external medium. The Kossovich number is the analogue of the heat transfer Biot number; the Posnov number is the analogue of the mass transfer Biot number. The Pn number influences only the mass-transfer potential field, while at the same time the Ko number influences the temperature field (Fig. 38). With increasing Pn and Ko numbers the dimensionless (average and local) mass and temperature potentials decrease linearly. This linearity applies at all values of Fo. With decreasing Pn numbers the rate of drying increases and the process proceeds at lower temperatures of the material — which may be a desirable condition. Calculations show that a decrease in the moisture content results from a sharp decrease of the Pn number. (See Fig. 37.) Such behavior has been verified by experiments. An increase of the Kossovich number results in an increase of $\bar{\theta}$ and a decrease of $\bar{T}$. The dependence of $\bar{T}$ and $\bar{\theta}$ on these internal heat and mass transfer parameters is just opposite to their dependence on the external heat and mass exchange parameters.

The nature of the dependence of heat and mass transfer on the Bi_m and Pn numbers underlies the method of handling experimental drying-out data worked out by Lebedev. P. D. Lebedev demonstrated



a

b

Fig. 38. Dependence of transfer potentials:
a — on Pn, and b — on Ko.

that for any drying method, the quantity $\frac{d\overline{\Theta}}{d\,\mathrm{Fo}}$ can be uniquely determined by two parameters: the ratio $\frac{\mathrm{Bi}_m}{\mathrm{Pn}}$ and the amount of moisture removed $(1 - \bar{u})$. It was shown above that surface mass transfer is characterized by the Bi_m, number, and internal mass transfer by the Pn number. Consequently, the two numbers completely characterize mass transfer in any given body.

Thus, Θ and $\frac{d\Theta}{d\,\mathrm{Fo}}$, the quantities that define mass transfer processes, should be completely described by their dependence on the Bi_m and Pn numbers. This dependence should hold equally well for different methods of drying, because the mechanism of heat and mass transfer remain the same.

From the foregoing analysis it is to be expected that the Bi_q and Ko numbers completely characterize heat transfer in any given body, since surface heat transfer is characterized by the former and internal heat transfer by the latter. The roles played by Bi_q, Ko, and Bi_q/Ko in heat transfer should be analogous to those of Bi_m, Pn, and Bi_m/Pn in mass transfer.

The role of the phase transformation number ε in heat and mass transfer is like that of the Kossovich number. The difference between these two numbers lies in the magnitude of their effect on thermal characteristics: the effect of ε is less pronounced than that of Ko. Similarities in the effects of ε and Ko on heat and mass transfer, as revealed by graphical representation and analysis of general solutions, indicate the existence of a functional dependence of the one dimensionless number on the other.

In this connection, it is of interest to assess the effect of the term describing phase transition (internal vaporization) $\frac{c'_\mathrm{T}\varepsilon\rho}{c_q}\frac{\partial\theta}{\partial\tau}$ see equation (I) — on the distributions of heat and mass transfer potentials and also on their rates of change. The influence of this term can be evaluated by comparing the results of calculation based on the solutions of the system of equations (I) to (VI) with both negative and positive signs for $\frac{c'_\mathrm{T}\varepsilon\rho}{c_q}\frac{\partial\theta}{\partial\tau}$. In this case the difference in the results amounts to twice the effect of the phase-transition term. Such calculations show the doubled effect of the phase-transition term upon the individual parameters to be as follows:

Table 31

For the dependence on the following dimensionless numbers	Average arithmetic % effect			
	$\overline{T}$	$\overline{\theta}$	$\frac{d\overline{T}}{d\,\mathrm{Fo}}$	$\frac{d\overline{\theta}}{d\,\mathrm{Fo}}$
Lu	20 (13 — 25)	4 (1 — 6)	2 (— 4 — + 7)	10 (3—16)
Bi_q	15 (12 — 21)	4 (4)	5 (2—7)	9
Bi_m	20 (19 — 21)	4 (3 — 6)	8 (4 — 12)	9 (8 — 11)
Pn	21 (24 — 18)	4 (0 — 7)	6 (— 8 — 22)	11 (5 — 16)
Ko	21 (6 — 36)	3 (— 2 — + 8)	6 (1—13)	12 (8 — 16)
Average effect	19,4	3,8	5,4	10,2

Thus, the most important effect exerted by this term is on the temperature field; there is almost no effect on the mass potential field (2%) and the heating rate of the material (3%); the average effect on all parameters is approximately 10%. The effect of this term on the dimensionless drying rate is also small; its numerical value is only 5%.

In cases where there is heat and mass transfer in an external medium with a time-dependent temperature, the general solution will contain the Predvoditelev number. The latter characterizes the rate of change of temperature of the medium. Let us analyze briefly the effect of the Pd number on heat and mass transfer when the temperature of the external medium decreases linearly with time.

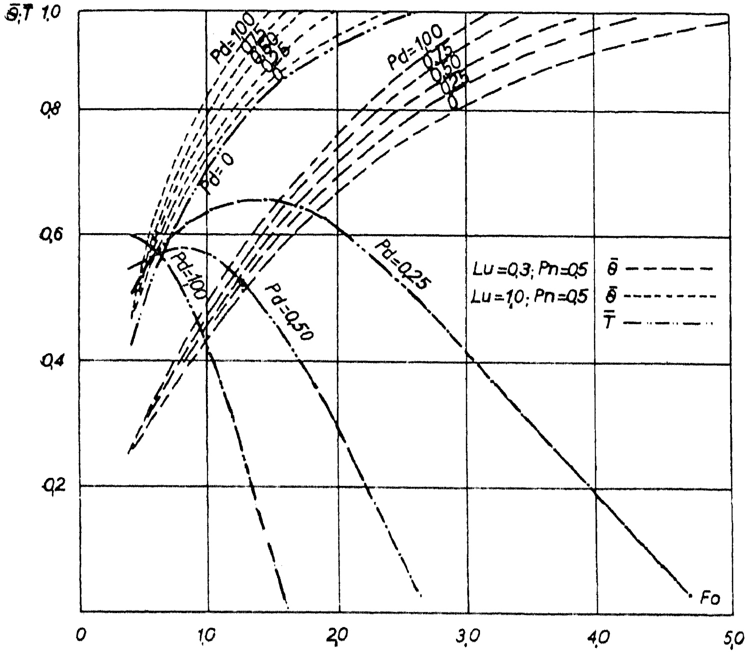


Fig. 39. Transfer potentials as a function of Pd and Fo.

The dimensionless temperature, when plotted as a function of Fo, initially increases, reaches a maximum and then decreases, (Fig. 39). The higher the values of Pd the sooner (i.e., at lower values of Fo and $\bar{\theta}$) the maximum temperature of the material is reached.

Until the maximum temperature is reached the surface temperature increases at a much higher rate than the temperature in the center of the material. After the maximum temperature is reached, the surface layer begins to cool because the temperature of the external medium at that time falls below the temperature of the material. However, because $t_{\text{п}} > t_{\text{ц}}$, heating of the center of the material continues. As soon as $t_{\text{ц}} \approx t_{\text{п}}$ all layers of the material undergo cooling. Knowledge of the exact time at which the maximum temperature is reached, and of the absolute value of the maximum temperature, is important for the correct design of any drying process. In order to prevent condensation from the external medium on the material, the drying process should be distontinued at $\text{Fo}_{\text{T}_{max}}$.

The heating rate decreases linearly with increasing Pd. The changes in the mass-transfer potential and heating rate with the Predvoditelev number are insignificant: with increasing Pd and Fo both the mass-transfer potential and the drying rate increase. Thus, at Fo = 0.4 the increase in the drying rate, when Pd increases from 0.25 to 1.0, amounts to 8.5%; and the change in the mass potential, $(\bar{\theta})$ is 3.5%. The corresponding values at Fo = 0.8 are 11.8% and 5.3%, respectively. With increasing values of Lu, the dependence of $\bar{\theta}$ on Pd remains as indicated. This dependence however occurs at higher values of $\bar{\theta}$.

The initial distributions of temperature and mass-transfer potential influence the heat and mass transfer only up to a certain value of Fo. Thus, beyond Fo ≥ 1.0) the value of W can be neglected for all practical purposes. The value of V, however, should be taken into account up to the values of $\bar{\theta}_1 = 0.7$ and Fo = 2.5.

In conclusion we shall consider the influence of the configuration on heat and mass transfer. Figure 40 shows that as the shape factor Γ increases, the average temperature of the body decreases (at constant mass concentration). The drying rate throughout the entire process is considerably higher for a sphere than for an infinite plate. The lowering of the drying and heating rates with increase in $\bar{\theta}$ is more pronounced for bodies with larger values of Γ. Thus, an increase in Γ causes a considerable rise in the drying process and lowers the average temperature of the material. Therefore, it can be concluded that a sphere can be dried more efficiently than a cylinder, and a cylinder can be dried more easily than a plate.

The heat and mass calculations can be considerably simplified by neglecting some of the dimensionless numbers. The simplified dimensionless equations for average heat and mass transfer variables can be presented as follows:

$$\bar{T} = \bar{T}\left(\text{Fo}, \text{Lu}, \frac{\text{Bi}_q}{\text{Ko}}, \varepsilon, \text{Pd}, V, W\right),$$

$$\bar{\Theta} = \bar{\Theta}\left(\text{Fo}, \text{Lu}, \frac{\text{Bi}_m}{\text{Pn}}, \text{Pd}, V\right),$$

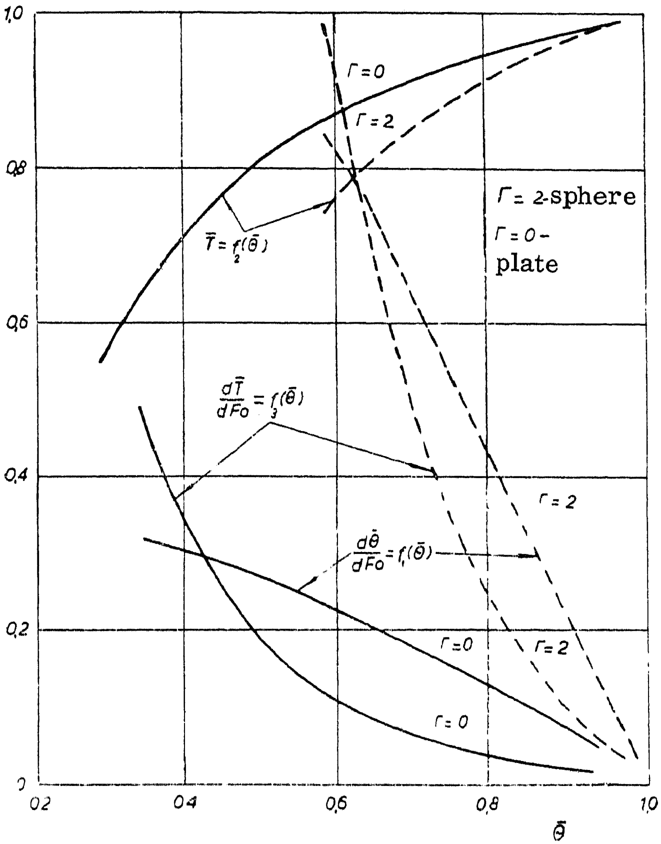


Fig. 40. The influence of the shape factor on internal heat and mass transfer.

$$\frac{d\bar{T}}{d\mathrm{Fo}} = \varphi_1\left(\mathrm{Fo}, \mathrm{Lu}, \frac{\mathrm{Bi}_q}{\mathrm{Ko}}, \varepsilon, \mathrm{Pd}, V, W\right),$$

$$\frac{d\bar{\Theta}}{d\mathrm{Fo}} = \varphi_2\left(\mathrm{Fo}, \mathrm{Lu}, \frac{\mathrm{Bi}_m}{\mathrm{Pn}}, \mathrm{Pd}, V\right).$$

In cases where the temperature of the external medium is constant Pd = 0. With uniform initial temperature distribution and mass transfer potential, V = 0 and W = 0. In cases where the heat and mass transfer processes are stationary, $\mathrm{Fo} = \infty$

7. HEAT AND MASS TRANSFER IN A LAYER OF MOIST MATERIAL

The system of differential equations (I) to (VI) describes heat and mass transfer within a single body. It is of interest to generalize this system of equations to that case in which heat and mass transfer processes occur within a layer. To begin with, such systems allow us to neglect a certain number of considerations (limitations) which are necessary in the complete analytic description of processes occurring within a layer. In the majority of work published on heat transfer and drying, the particles of the layer are considered to be very small and consequently the gradients of heat and mass transfer potentials in the particles are neglected. The layer of the moist material is regarded as a continuous medium with continuously and uniformly changing temperature and mass potential. For the majority of actual processes that occur in a layer, the size of the elements of the layer is not insignificant in comparison with the dimensions of the layer itself. Moreover, if the temperature of the external medium is high, then the potential gradients in the material will be large. In such cases, it is incorrect to regard the layer as a continuous medium. The generalization of the system of equations (I) to (VI) ought to provide a connection between the surface temperature of the particle and the temperature of the external medium. Such a generalization at the same time will provide a more complete and more accurate accounting of the temperature change of the external medium. Therefore, this section can be considered as an extension of the material presented in Sections 2 and 3.

a) Statement of the Problem: The System of Differential Equations.

Let us consider a layer of moist material consisting of spherical particles of radius R. The external medium is a heat carrier moving through this layer in the direction η, at a constant rate v. The ambient temperature around the layer is $\overset{\circ}{t}_c$. The initial transfer potentials in the particles of the layer are t_0 and θ_0 (i.e., until the particles make contact with the heat carrier). The porosity of the latter is P.

If radiation within the layer and heat conduction (by physical contact) among the particles of the layer are neglected, the system of transfer equations may be written in the following form:

$$\frac{\partial\,[rt(r,\tau)]}{\partial\tau} = a_q\,\frac{\partial^2\,[rt(r,\tau)]}{\partial r^2} + \frac{\varepsilon\rho c_{\mathrm{T}}'}{c_q}\,\frac{\partial\,[r\theta(r,\tau)]}{\partial\tau},$$

$$\frac{\partial\,[r\theta(r,\tau)]}{\partial\tau} = a_m\,\frac{\partial^2\,[r\theta(r,\tau)]}{\partial r^2} + a_m\,\delta\,\frac{\partial^2\,[rt(r,\tau)]}{\partial r^2},$$

$$-\lambda_q \frac{\partial t(R,\tau)}{\partial r} + \alpha\,[t_c(\eta,\tau) - t(R,\tau)] - (1-\varepsilon)\,\rho\, q_m = 0, \qquad (7.1)$$

$$\lambda_m \left[\frac{\partial \theta(R,\tau)}{\partial r} + \delta\,\frac{\partial t(R,\tau)}{\partial r}\right] + q_m = 0,$$

$$\frac{\partial t_c(\eta,\tau)}{\partial \tau} + v\,\frac{\partial t_c(\eta,\tau)}{\partial \eta} + \frac{\alpha F}{c_p \gamma_p p}\,[t_c(\eta,\tau) - t(R,\tau)] = 0,$$

$$\frac{\partial t(0,\tau)}{\partial r} = \frac{\partial \theta(0,\tau)}{\partial r} = 0$$

with

$$\eta = 0 \quad t_c = t_c^{\circ} = \text{const},$$

$$\eta = v\tau \quad t = t_0 = \text{const},\ \theta = \theta_0 = \text{const}.$$

b) Dimensionless Equations for Heat and Mass Transfer in a Layer.

From the system of equations (7.1) we obtain new dimensionless numbers:

The dimensionless time Ho, which is defined by

$$\text{Ho} = \frac{v\tau}{R};$$

the Margulis number, which is a measure of the ratio of heat transferred to the particle to the heat given up by the external medium (heat carrier) while moving through the layer,

$$\text{Mr} = \frac{\alpha}{v c_p \gamma_p};$$

and the Miniovich number, which consists of dimensions characterizing the geometry of the layer,

$$\text{Mn} = \frac{FR}{p}.$$

Since Ho = FoPe, where $\text{Pe} = \dfrac{vR}{a_q}$ is the Peclet number, the dimensionless equations for the layer of the moist material can be written in the following form:

$$T = T\,(\text{Fo}, \text{Lu}, \text{Bi}_q, \text{Bi}_m, \text{Ko}, \text{Pn}, \varepsilon, \text{Pe}, \text{Mr}, \text{Mn}, W, V, x/R, \eta/R), \qquad (7.2)$$

$$\theta = \theta\,(\text{Fo}, \text{Lu}, \text{Bi}_q, \text{Bi}_m, \text{Ko}, \text{Pn}, \varepsilon, \text{Pe}, \text{Mr}, \text{Mn}, W, V, x/R, \eta/R). \qquad (7.3)$$

The number of independent dimensionless parameters is considerably smaller than the number of dimensionless parameters appearing in (7.2) and (7.3), because those that do appear are interrelated.

c) Solutions for the Temperature Distribution in the Particles and in the External Medium (Heat Carrier) for $\rho = 0$.

$$T = \frac{t(r, \tau) - t_0}{t_c^{\circ} - t_0} = 1 - \sum_{n=1}^{\infty} A_n \left(1 - \mathrm{MrMn}\,\frac{\eta}{R}\right) \times$$

$$\times \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} \exp\left[-\mu_n^2 K_3 \left(\mathrm{PeFo} - \frac{\eta}{R}\right)\right] -$$

$$- \sum_{n=1}^{\infty} A_n^2 \frac{\sin \mu_n}{\mu_n} \left\{ \left[\frac{\mu_n^2}{\mu_n^2 + \mathrm{Bi}_q (\mathrm{Bi}_q - 1)} + \right.\right.$$

$$\left. + \mu_n^2 K_3 \left(\mathrm{PeFo} - \frac{\eta}{R}\right) + \frac{\mathrm{Bi}_q}{2}\right] \times$$

$$\left. \times \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} - \cos \mu_n \frac{r}{R} \right\} \mathrm{MrMn}\,\frac{\eta}{R} \exp \times$$

$$\times \left[-\mu_n^2 K_3 \left(\mathrm{Pe\,Fo} - \frac{\eta}{R}\right)\right] + \ldots$$

$$T_c = \frac{t_c(\eta, \tau) - t_0}{t_c^0 - t_0} = 1 - \sum_{n=1}^{\infty} A_n \mathrm{MrMn}\,\frac{\eta}{R} \exp \times$$

$$\times \left[-\mu_n^2 K_3 \left(\mathrm{PeFo} - \frac{\eta}{R}\right)\right] + \ldots$$

Here

$$A_n = \frac{2(\sin \mu_n - \mu_n \cos \mu_n)}{\mu_n - \sin \mu_n \cos \mu_n} = \frac{2\,\mathrm{Bi}_q \sqrt{\mu_n^2 + (\mathrm{Bi}_q - 1)^2}}{\mu_n^2 + \mathrm{Bi}_q (\mathrm{Bi}_q - 1)},$$

$$K_3 = -\frac{\mathrm{Mr}}{\mathrm{Bi}_q} \frac{c_p \gamma_p}{c_q \gamma_0}$$

and μ_n are the roots of the characteristic equation

$$\mathrm{tg}\, \mu = - \frac{\mu}{\mathrm{Bi}_q - 1}.$$

CHAPTER III

Non-Stationary Fields of Heat and Mass Transfer Potentials with Boundary Conditions of the First Kind

In some thermal and chemical systems, for which heat and mass transfer coefficients are to be determined, it is necessary to maintain constant heat- and mass-transfer potentials on the surface of the system. The boundary conditions for such processes are referred to in mathematics as being of the first kind. The solutions of the system of differential equations (I) and (II) for heat and mass transfer in this case can be obtained from the solutions of the corresponding problems contained in the previous chapter by putting the Bi_q and Bi_m numbers equal to infinite. The solutions of equations (I) and (II) with constant values of the average mass transfer potential,

$$\frac{\Gamma+1}{R^{\Gamma+1}}\int_0^R \xi^{\Gamma}\theta(\xi,\tau)\,d\xi=\theta_0=\mathrm{const} \tag{VII}$$

and with constant temperature on the surface of the body,

$$t(R,\tau)=0, \tag{VIII}$$

are given below. The results given in Section 1 were obtained on the basis of solutions presented by S. P. Kuznetsov; the solutions given in Section 2d were obtained by E. I. Sizyakova, and in Section 2a and 2c by S. P. Kuznetsov.

1. NON-STATIONARY FIELDS OF HEAT AND MASS TRANSFER POTENTIALS WITH UNIFORM INITIAL MASS CONCENTRATION.

a) Solution for an Infinite Plate ($\xi = x,\ \Gamma = 0,\ t(x,0) = t_0 = \mathrm{const},$ $\theta(x,0) = \theta_0 = \mathrm{const}$).

The general transform solutions of (I), (II) can be written in the following form: (See Chapter II, Section 1):

$$t_L(x, s) = \frac{t_0}{s} + B_1 \mathrm{ch} \sqrt{\frac{s}{a_q}}\, \nu_1 x + B_2 \mathrm{ch} \sqrt{\frac{s}{a_q}}\, \nu_2 x, \tag{1.1}$$

$$\theta_L(x, s) = \frac{\theta_0}{s} + \frac{c}{\varepsilon \rho} B_1 (1 - \nu_1^2) \mathrm{ch} \sqrt{\frac{s}{a_q}}\, \nu_1 x +$$

$$+ \frac{c}{\varepsilon \rho} B_2 (1 - \nu_2^2) \mathrm{ch} \sqrt{\frac{s}{a_q}}\, \nu_2 x, \tag{1.2}$$

where

$$\nu_j^2 = \frac{1}{2}\left[\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right) + (-1)^j \times\right.$$

$$\left.\times \sqrt{\left(1 + \mathrm{Fe} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}}\,\right],$$

B_1 and B_2 are constant with respect to x. The latter will be determined from boundary conditions (VII) and (VIII), which in terms of transforms can be represented by the following:

$$\frac{1}{R}\int_0^R \theta(x, s)\, dx = \frac{\theta_0}{s}, \tag{1.3}$$

$$t_L(R, s) = 0. \tag{1.4}$$

In order that solutions (1.1), and (1.2) satisfy conditions (1.3) and (1.4) we have

$$\nu_2 (1 - \nu_1^2) B_1 \mathrm{sh} \sqrt{\frac{s}{a_q}}\, \nu_1 R + (1 - \nu_2^2) \nu_1 B_2 \mathrm{sh} \times$$

$$\times \sqrt{\frac{s}{a_q}}\, \nu_2 R = 0, \tag{1.5}$$

$$B_1 \mathrm{ch} \sqrt{\frac{s}{a_q}}\, \nu_1 R + B_2 \mathrm{ch} \sqrt{\frac{s}{a_q}}\, \nu_2 R + \frac{t_0}{s} = 0. \tag{1.6}$$

B_1 is obtained from equation (1.6) and substituted into equation (1.5):

$$B_2 = \nu_2 (1 - \nu_1^2) t_0 \mathrm{sh} \sqrt{\frac{s}{a_q}}\, \nu_1 R \times$$

$$\times s\left[\nu_1(1-\nu_2^2)\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R-\right.$$

$$\left.-\nu_2(1-\nu_1^2)\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R\right]^{-1}.$$

Then,

$$B_1=\nu_1(1-\nu_2^2)\,t_0\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\times$$

$$\times s\left[\nu_1(1-\nu_2^2)\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R-\right.$$

$$\left.-\nu_2(1-\nu_1^2)\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R\right]^{-1}$$

Consequently, the transforms for the transfer potentials take the following form:

$$t_L(x,s)-\frac{t_0}{s}=-t_0\,\frac{K_4\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x-\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x}{s\left[K_4\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R-\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R\right]}=-t_0\,\frac{\Phi_1(s)}{\psi(s)}, \quad (1.7)$$

$$\frac{\Theta_0}{s}-\Theta_L(x,s)=\frac{c}{\varepsilon\rho}(1-\nu_2^2)\,t_0\,\frac{\frac{\nu_1}{\nu_2}\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\times\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 x-\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 x}{s\left[K_4\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\times\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R-\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R\right]}=$$

$$=\frac{ct_0}{\varepsilon\rho}(1-\nu_2^2)\,\frac{\Phi_2(s)}{\psi(s)}, \quad (1.8)$$

where

$$K_4=\frac{\nu_1}{\nu_2}\left(\frac{1-\nu_2^2}{1-\nu_1^2}\right).$$

In order to obtain the inversions of $t(x, \tau)$ and $\theta(x, \tau)$ we shall utilize the expansion theorem.

The expansion theorem is applicable because expressions (1.7) and (1.8) can be represented as ratios of generalized polynomials with respect to s, where the exponent of the polynomials in the numerator is smaller by one than the exponent of the polynomial in the denominator.

We shall now find the roots of $\psi(s)$ equal to zero. Here we have

1) zeroth root, $s = 0$,

2) roots satisfying the following equation:

$$K_4 \operatorname{sh} \sqrt{\frac{s}{a_q}} \nu_2 R \operatorname{ch} \sqrt{\frac{s}{a_q}} \nu_1 R - \operatorname{sh} \sqrt{\frac{s}{a_q}} \nu_1 R \operatorname{ch} \sqrt{\frac{s}{a_q}} \nu_2 R = 0.$$

The above can be represented as follows:

$$K_4 \sin N\mu \cos\mu - \sin\mu \cos N\mu = 0, \tag{1.9}$$

where

$$\mu = i\sqrt{\frac{s}{a_q}} \nu_1 R, \quad N = \frac{\nu_2}{\nu_1}.$$

Equation (1.9) has an infinite number of real simple roots, which are the common roots of the system of equations

$$\left.\begin{array}{l} \sin\mu = 0 \\ \sin N\mu = 0 \end{array}\right\}, \tag{1.10}$$

and the roots of the following equation:

$$\operatorname{ctg} N\mu - K_4 \operatorname{ctg}\mu = 0. \tag{1.11}$$

The system of equations (1.10) contains general roots only in that case when N is a rational fraction. The numerical values of the roots of equation (1.11) can be obtained by a graphical method. For equation (1.7) the expansion theorem will be written in the following form:

$$t(x, \tau) = -t_0 \left[\lim_{s\to 0} \frac{\Phi_1(s)}{\psi'(s)} + \sum_{l=1}^{\infty} \frac{\Phi_1(s_l)}{\psi'(s_l)} e^{s_l \tau} + \right.$$
$$\left. + \sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi'(s_n)} e^{s_n \tau} \right]. \tag{1.12}$$

In order to determine the first term of equation (1.12) we shall represent the ratio $\Phi_1 \dfrac{(s)}{\psi(s)}$ as

$$\frac{\Phi_1(s)}{\psi(s)}=\frac{K_4\left[1+\frac{1}{2!}\left(\sqrt{\frac{s}{a_q}}\nu_1 x\right)^2+\ldots\right]-\left[1+\frac{1}{2!}\left(\sqrt{\frac{s}{a_q}}\nu_2 x\right)^2+\ldots\right]\times\dfrac{\operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_1 R}{\operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_2 R}}{s\left\{K_4\left[1+\frac{1}{2!}\left(\sqrt{\frac{s}{a_q}}\nu_1 R\right)^2+\ldots\right]-\left[1+\frac{1}{2!}\left(\sqrt{\frac{s}{a_q}}\nu_2 R\right)^2+\ldots\right]\times\dfrac{\operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_1 R}{\operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_2 R}\right\}},\qquad(1.13)$$

where

$$\frac{\operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_1 R}{\operatorname{sh}\sqrt{\frac{s}{a_q}}\nu_2 R}=\frac{1+\frac{1}{3!}\left(\sqrt{\frac{s}{a_q}}\nu_1 R\right)^2+\ldots}{1+\frac{1}{3!}\left(\sqrt{\frac{s}{a_q}}\nu_2 R\right)^2+\ldots}\,\frac{\nu_1}{\nu_2}.$$

Differentiating the denominator of equation (1.13) with respect to s and obtaining the limits of this expression as $s\to 0$, we get

$$\lim_{s\to 0}\frac{\Phi_1(s)}{\psi'(s)}=1$$

In the second term of equation (1.12) the summation is conducted with respect to the general roots of the system (1.10). Let us denote the general roots by $\mu_l=\nu_1 l\pi$, with $l=1, 2, 3\ldots$ Then, substituting trigonometrical functions for the hyperbolic functions, we get

$$\Phi_1(s_l)=-i\left(K_4\sin N\mu_l\cos\mu_l\frac{x}{R}-\sin\mu_l\cos N\mu_l\frac{x}{R}\right)=$$

$$=-i\left(K_4\sin\nu_2 l\pi\cos\nu_1 l\pi\frac{x}{R}-\right.$$

$$\left.-\sin\nu_1 l\pi\cos\nu_2 l\pi\frac{x}{R}\right)=0,$$

and

$$\psi'(s_l) = -i\left(K_4 \frac{N\mu_l}{2} \cos N\mu_l \cos \mu_l - \frac{\mu_l}{2} \cos \mu_l \cos N\mu_l\right) =$$

$$= -i\frac{\mu_l}{2}(K_4 N - 1) \cos N\mu_l \cos \mu_l \neq 0.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{\Phi_1(s_l)}{\psi'(s_l)} \exp(s_l \tau) = 0.$$

In the third term, the summation is conducted with respect to the positive roots of equation (1.11) and, consequently,

$$\sum_{n=1}^{\infty} \frac{\Phi_1(s_n)}{\psi'(s_n)} e^{s_n\tau} =$$

$$= 2\sum_{n=1}^{\infty} \frac{K_4 \sin N\mu_n \cos \mu_n \frac{x}{R} - \sin \mu_n \cos N\mu_n \frac{x}{R}}{\mu_n\left[(NK_4 - 1) \cos \mu_n \cos N\mu_n + (N - K_4) \sin \mu_n \sin N\mu_n\right]} \exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right).$$

Substituting for the appropriate terms in equation (1.12) we get

$$\frac{t(x,\tau)}{t_0} =$$

$$= -\sum_{n=1}^{\infty}\left(C_{1n} \cos \mu_n \frac{x}{R} - C_{2n} \cos N\mu_n \frac{x}{R}\right) \exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right), \qquad (1.14)$$

where

$$C_{1n} = \frac{2K_4 \sin N\mu_n}{\mu_n\left[(NK_4 - 1) \cos \mu_n \cos N\mu_n + (N - K_4) \sin \mu_n \sin N\mu_n\right]},$$

$$C_{2n} = \frac{2 \sin \mu_n}{\mu_n\left[(NK_4 - 1) \cos \mu_n \cos N\mu_n + (N - K_4) \sin \mu_n \sin N\mu_n\right]},$$

μ_n are the roots of the characteristic equation (1.11).

We shall now determine the distribution of mass transfer potential in an infinite plate. Since the roots of the denominator of equation (1.8) remain as in the previous case, the expansion theorem can be written as follows:

$$\theta_0-\theta(x,\tau)=\frac{c}{\varepsilon\rho}t_0(1-\nu_2^2)\left[\lim_{s\to 0}\frac{\Phi_2(s)}{\psi'(s)}+\right.$$
$$\left.+\sum_{l=1}^{\infty}\frac{\Phi_2(s_l)}{\psi'(s_l)}e^{s_l\tau}+\sum_{n=1}^{\infty}\frac{\Phi_2(s_n)}{\psi'(s_n)}e^{s_n\tau}\right]. \qquad (1.15)$$

By means of a technique analogous to the method employed in determining the first two terms of equation (1.12) it can be shown that the corresponding terms of equation (1.15) equal zero. The third term yields

$$\sum_{n=1}^{\infty}\frac{\Phi_2(s_n)}{\psi'(s_n)}e^{s_n\tau}=$$
$$=2\sum_{n=1}^{\infty}\frac{\left[\frac{1}{N}\sin N\mu_n\cos\mu_n\frac{x}{R}-\sin\mu_n\cos N\mu_n\frac{x}{R}\right]\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right)}{\mu_n\left[(NK_4-1)\cos\mu_n\cos N\mu_n+(N-K_4)\sin\mu_n\sin N\mu_n\right]}.$$

Thus, for the mass transfer potential field we finally get

$$\frac{\theta_0-\theta(x,\tau)}{\theta_0}=$$
$$=\frac{1-\nu_2^2}{\varepsilon\mathrm{Ko}_1}\sum_{n=1}^{\infty}\left(C_{1n}^{*}\cos\mu_n\frac{x}{R}-C_{2n}^{*}\cos N\mu_n\frac{x}{R}\right)\times$$
$$\times\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right), \qquad (1.16)$$

where

$$C_{1n}^{*}=\frac{1}{NK_4}C_{1n},\quad C_{2n}^{*}=C_{2n},$$

$\mathrm{Ko}_1=\frac{\rho c'_T\theta_0}{c_q t_c}$ is the Kossovich number,

and μ_n are the roots of characteristic equation (1.11).

The average values of transfer potentials are determined by the following relationships:

$$\bar{t}(\tau)=\frac{1}{R}\int_0^R t(x,\tau)\,dx,\qquad \bar{\theta}(\tau)=\frac{1}{R}\int_0^R\theta(x,\tau)\,dx.$$

The average dimensionless temperature is

$$\frac{\bar{t}(\tau)}{t_0} = -\sum_{n=1}^{\infty} D_n \exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right) \qquad (1.17)$$

and the average dimensionless mass-transfer potential is

$$\frac{\theta_0 - \bar{\theta}(\tau)}{\theta_0} = \frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\sum_{n=1}^{\infty} D_n^* \exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right), \qquad (1.18)$$

where

$$D_n = C_{1n}\frac{\sin\mu_n}{\mu_n} - C_{2n}\frac{\sin N\mu_n}{N\mu_n},$$

$$D_n^* = C_{1n}^*\frac{\sin\mu_n}{\mu_n} - C_{2n}^*\frac{\sin N\mu_n}{N\mu_n}.$$

The local and average dimensionless rates of change of the transfer potentials will be obtained by differentiation of equations (1.14) and (1.16) to (1.18) with respect to the dimensionless time, i.e.,

$$\frac{d\left[\frac{t(x,\tau)}{t_0}\right]}{d\,\mathrm{Fo}} =$$

$$= \sum_{n=1}^{\infty}\frac{\mu_n^2}{\nu_2^1}\left(C_{1n}\cos\mu_n\frac{x}{R} - C_{2n}\cos N\mu_n\frac{x}{R}\right)\times$$

$$\times\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right),$$

$$\frac{d\left[\frac{\bar{t}(\tau)}{t_0}\right]}{d\,\mathrm{Fo}} = \sum_{n=1}^{\infty}\frac{\mu_n^2}{\nu_1^2} D_n \exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right),$$

$$\frac{d\left[\frac{\theta(x,\tau)}{\theta_0}\right]}{d\,\mathrm{Fo}} =$$

$$= \frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\sum_{n=1}^{\infty}\frac{\mu_n^2}{\nu_1^2}\left(C_{1n}^*\cos\mu_n\frac{x}{R} - C_{2n}^*\cos N\mu_n\frac{x}{R}\right)\times$$

$$\times\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right),$$

$$\frac{d\left[\frac{\overline{\theta}(\tau)}{\theta_0}\right]}{d\mathrm{Fo}}=\frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\sum_{n=1}^{\infty}\frac{\mu_n^2}{\nu_1^2}D_n^*\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right).$$

b) Solution for a Sphere

$$(\xi=r,\quad \Gamma=2,\quad t(r,0)=t_0=\mathrm{const},\quad \theta(r,0)=\theta_0=\mathrm{const}).$$

The general transform solutions of (I) and (II) can be given as (see Chapter II, Section 1b):

$$t_L(r,s)=\frac{t_0}{s}+\frac{B_1}{r}\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r+$$

$$+\frac{B_2}{r}\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r, \tag{1.19}$$

$$\theta_L(r,s)=\frac{\theta_0}{s}+\frac{c}{\varepsilon\rho}(1-\nu_1^2)\frac{B_1}{r}\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r+$$

$$+\frac{c}{\varepsilon\rho}(1-\nu_2^2)\frac{B_2}{r}\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r. \tag{1.20}$$

In order that equations (1.19) and (1.20) satisfy boundary conditions (VII) and (VIII) we have

$$B_1\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R+B_2\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R+\frac{R\,t_0}{s}=0, \tag{1.21}$$

$$(1-\nu_1^2)B_1\frac{\sqrt{\frac{s}{a_q}}\,\nu_1 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R-\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R}{\left(\sqrt{\frac{s}{a_q}}\,\nu_1 R\right)^2}+$$

$$+(1-\nu_2^2)B_2\frac{\sqrt{\frac{s}{a_q}}\,\nu_2 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R-\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R}{\left(\sqrt{\frac{s}{a_q}}\,\nu_2 R\right)^2}=0. \tag{1.22}$$

Solving equations (1.21) and (1.22) simultaneously for B_1 and B_2, we get

$$B_1=-\frac{K_4}{N}R\,t_0\left(\sqrt{\frac{s}{a_q}}\,\nu_2 R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R-\mathrm{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\right)\times$$

$$\times s\left[\frac{K_4}{N}\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\times\right.$$

$$\times\left(\sqrt{\frac{s}{a_q}}\,\nu_2 R\operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\right)-$$

$$-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\left(\sqrt{\frac{s}{a_q}}\,\nu_1 R\times\right.$$

$$\left.\left.\times\operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\right)\right]^{-1},$$

$$B_2=R\,t_0\left(\sqrt{\frac{s}{a_q}}\,\nu_1 R\operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\right)\times$$

$$\times s\left[\frac{K_4}{N}\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\times\right.$$

$$\times\left(\sqrt{\frac{s}{a_q}}\,\nu_2 R\operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\right)-$$

$$-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\left(\sqrt{\frac{s}{a_q}}\,\nu_1 R\times\right.$$

$$\left.\left.\times\operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\right)\right]^{-1}.$$

With the above and after some operations, the transform solutions for the distribution of heat and mass transfer potentials can be written as follows:

$$t_L(r,s)-\frac{t_0}{s}=-\frac{R\,t_0}{r}\times$$

$$\times\frac{\dfrac{K_4}{N}\left(\sqrt{\dfrac{s}{a_q}}\,\nu_2 R\operatorname{ch}\sqrt{\dfrac{s}{a_q}}\,\nu_2 R-\operatorname{sh}\sqrt{\dfrac{s}{a_q}}\,\nu_2 R\right)\times}{s\left[\dfrac{K_4}{N}\left(\sqrt{\dfrac{s}{a_q}}\,\nu_2 R\operatorname{ch}\sqrt{\dfrac{s}{a_q}}\,\nu_2 R-\operatorname{sh}\sqrt{\dfrac{s}{a_q}}\,\nu_2 R\right)\times\right.}$$

$$\frac{\times\operatorname{sh}\sqrt{\dfrac{s}{a_q}}\,\nu_1 r-\left(\sqrt{\dfrac{s}{a_q}}\,\nu_1 R\operatorname{ch}\sqrt{\dfrac{s}{a_q}}\,\nu_1 R-\right.}{\times\operatorname{sh}\sqrt{\dfrac{s}{a_q}}\,\nu_1 R-\left(\sqrt{\dfrac{s}{a_q}}\,\nu_1 R\operatorname{ch}\sqrt{\dfrac{s}{a_q}}\,\nu_1 R-\right.}$$

$$\frac{-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\Big)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r}{-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\Big)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\Big]} = -\frac{R t_0}{r}\,\frac{\Phi_1(s)}{\psi(s)}, \qquad (1.23)$$

$$\frac{\theta_0}{s} - \theta_L(r, s) = \frac{c}{\varepsilon\rho}\,\frac{R t_0}{r}\,\frac{1-\nu_2^2}{N^2}\times$$

$$\times\frac{\Big(\sqrt{\frac{s}{a_q}}\,\nu_2 R \operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R - \operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\Big)\times}{s\Big[\frac{K_4}{N}\Big(\sqrt{\frac{s}{a_q}}\,\nu_2 R \operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R - \operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\Big)\times}$$

$$\frac{\times\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r - N^2\Big(\sqrt{\frac{s}{a_q}}\,\nu_1 R \operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R -}{\times\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R - \Big(\sqrt{\frac{s}{a_q}}\,\nu_1 R \operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R -}$$

$$\frac{-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\Big)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 r}{-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\Big)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\Big]} =$$

$$= \frac{c}{\varepsilon\rho}\,\frac{R t_0}{r}\,\frac{(1-\nu_2^2)}{N^2}\,\frac{\Phi_2(s)}{\psi(s)}, \qquad (1.24)$$

where the values of $\Phi_1(s)$, $\Phi_2(s)$ and $\psi(s)$ follow directly from equations (1.23 and (1.24). K_4 and N were determined above.

In order to obtain the inversions $t(r, \tau)$ and $\theta(r, \tau)$, we shall use the expansion theorem. The ratios $\frac{\Phi_1(s)}{\psi(s)}$ and $\frac{\Phi_2(s)}{\psi(s)}$ can be considered ratios of generalized polynomials with respect to s, to which the theorem of expansion can be applied. The common denominator $\psi(s)$ has the following roots:

a — zeroth root, $s = 0$,

b — the roots of the equation

$$\frac{K_4}{N}\Big(\sqrt{\frac{s}{a_q}}\,\nu_2 R \operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_2 R - \operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R\Big)\times$$

$$\times\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R - \Big(\sqrt{\frac{s}{a_q}}\,\nu_1 R \operatorname{ch}\sqrt{\frac{s}{a_q}}\,\nu_1 R -$$

$$-\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R\Big)\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_2 R = 0,$$

which can also be rewritten in the following form:

$$\frac{K_4}{N}(\sin N\mu - N\mu\cos N\mu)\sin\mu - (\sin\mu - \mu\cos\mu)\,\sin N\mu = 0,$$

where

$$\mu = i\sqrt{\frac{s}{a_q}}\,\nu_1 R.$$

The above equation is satisfied by the common roots of the following system of equations:

$$\left.\begin{aligned}\sin\mu &= 0\\ \sin N\mu &= 0\end{aligned}\right\}, \tag{1.25}$$

(i.e., when N is a rational fraction), and by the roots of equations:

$$K_4\,\mu\,\mathrm{ctg}\,N\mu - \mu\,\mathrm{ctg}\,\mu - \frac{K_4}{N} + 1 = 0. \tag{1.26}$$

Applying the theorem of expansion to equation (1.23), we get

$$t(r,\tau) - t_0 = -\frac{R\,t_0}{r}\left[\lim_{s\to 0}\frac{\Phi_1(s)}{\psi'(s)} + \right.$$

$$\left. + \sum_{l=1}^{\infty}\frac{\Phi_1(s_l)}{\psi'(s_l)}\,e^{s_l\tau} + \sum_{n=1}^{\infty}\frac{\Phi_1(s_n)}{\psi'(s_n)}\,e^{s_n\tau}\right]. \tag{1.27}$$

Here,

s_l are the common roots of the system of equations (1.25), and

s_n are the roots of equation (1.26).

If s is set equal to zero in the ratio $\frac{\Phi_1(s)}{\psi'(s)}$ then equation (1.27) will be indeterminate. Therefore, we shall first expand the hyperbolic functions entering into $\Phi_1(s)$ and $\psi(s)$ as infinite series and then simplify to get

$$\frac{\Phi_1(s)}{\psi(s)} = \frac{\frac{K_4}{N}\left[\left(\frac{1}{2!} - \frac{1}{3!}\right) + \frac{s}{a_q}\,\nu_2^2 R^2\left(\frac{1}{4!} - \frac{1}{5!}\right) + \dots\right]\times}{s\left[\frac{K_4}{N}\left(\frac{1}{2!} - \frac{1}{3!}\right) + \frac{s}{a_q}\,\nu_2^2 R^2\left(\frac{1}{4!} - \frac{1}{5!}\right) + \dots\right]\times}$$

$$\frac{\times \nu_2^2 - \left[\left(\frac{1}{2!} - \frac{1}{3!}\right) + \frac{s}{a_q}\nu_1^2 R^2\left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots\right]\times}{\times \nu_2^2 - \left[\left(\frac{1}{2!} - \frac{1}{3!}\right) + \frac{s}{a_q}\nu_1^2 R^2\left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots\right]\times}$$

$$\frac{\times \nu_1^2 \dfrac{1 + \frac{1}{3!}\left(\sqrt{\frac{s}{a_q}}\,\nu_2\right)^2 r^2 + \ldots}{1 + \frac{1}{3!}\left(\sqrt{\frac{s}{a_q}}\,\nu_1\right)^2 r^2 + \ldots}}{\times \nu_1^2 \dfrac{1 + \frac{1}{3!}\left(\sqrt{\frac{s}{a_q}}\,\nu_2 R\right)^2 + \ldots}{1 + \frac{1}{3!}\left(\sqrt{\frac{s}{a_q}}\,\nu_1 R\right)^2 + \ldots}} \times$$

$$\times \frac{\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r}{\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R},$$

where the last term is

$$\frac{\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 r}{\operatorname{sh}\sqrt{\frac{s}{a_q}}\,\nu_1 R} = \frac{1 + \frac{1}{3!}\left(\sqrt{\frac{s}{a_q}}\,\nu_1 r\right)^2 + \ldots}{1 + \frac{1}{3!}\left(\sqrt{\frac{s}{a_q}}\,\nu_1 R\right)^2 + \ldots}\,\frac{r}{R}.$$

Differentiating the denominator of the ratio with respect to s and obtaining the limit of the fraction $\frac{\Phi_1(s)}{\psi'(s)}$ at $s \to 0$, we get for the first term of equation (1.27):

$$\lim_{s \to 0} \frac{\Phi_1(s)}{\psi'(s)} = \frac{r}{R}.$$

We shall now find the second term of equation (1.27). The common roots of the system of equations (1.25) are $\mu_l = \beta l \pi$, at $l = 1, 2, 3\ldots$, if $N = \frac{\nu_2}{\nu_1} = \frac{\alpha}{\beta}$ is a rational fraction (α and β are integers). It is easy to demonstrate that

$$\Phi_1(s_l) = \mu_l \cos\mu_l \,\sin N\mu_l \frac{r}{R} - \frac{K_4}{N} N\mu_l \,\cos N\mu_l \,\sin\mu_l \frac{r}{R},$$

$$\psi'(s_l) = \frac{1}{2}\,\mu_l N \mu_l \cos\mu_l \cos N\mu_l \left(1 - \frac{K_4}{N}\right).$$

because

$$\cos\mu_l = \cos\beta l\pi = (-1)^{\beta l} \text{ and } \cos N\mu_l = \cos\alpha l\pi = (-1)^{\alpha l},$$

then we finally get for the second term of equation (1.27)

$$\sum_{l=1}^{\infty} \frac{\Phi_1(s_l)}{\psi'(s_l)} \exp(s_l\tau) =$$

$$= 2\sum_{l=1}^{\infty} \frac{(-1)^{\beta l}\beta \sin\alpha l\pi \frac{r}{R} - \frac{K_4}{N}(-1)^{\alpha l}\alpha \sin\beta l\pi \frac{r}{R}}{\alpha\beta l\pi(-1)^{\beta l}(-1)^{\alpha l}\left(1 - \frac{K_4}{N}\right)} \times$$

$$\times \exp\left(-\frac{\beta^2 l^2 \pi^2 a_q \tau}{\nu_1^2 R^2}\right).$$

The third term of equation (1.27) is

$$\Phi_1(s_n) = \frac{K_4}{N}(\sin N\mu_n - N\mu_n \cos N\mu_n)\sin\mu_n \frac{r}{R} -$$

$$-(\sin\mu_n - \mu_n\cos\mu_n)\sin N\mu_n \frac{r}{R},$$

$$\psi'(s_n) = \frac{1}{2}\left[\mu_n \frac{K_4}{N}\cos\mu_n(\sin N\mu_n - N\mu_n\cos N\mu_n) -\right.$$

$$- N\mu_n \cos N\mu_n(\sin\mu_n - \mu_n\cos\mu_n) +$$

$$\left. + (NK_4 - 1)\mu_n^2 \sin\mu_n \sin N\mu_n\right].$$

Thus, the heat-transfer potential distribution in a sphere can be described as follows:

$$\frac{t(r,\tau)}{t_0} =$$

$$= -\left[\sum_{l=1}^{\infty}\left(E_{1l}\frac{\sin\alpha\pi l \frac{r}{R}}{\frac{r}{R}} - E_{2l}\frac{\sin\beta\pi l \frac{r}{R}}{\frac{r}{R}}\right)\times\right.$$

$$\times \exp\left(-\frac{\beta^2\pi^2 l^2}{\nu_1^2}\,\mathrm{Fo}\right) +$$

$$+\sum_{n=1}^{\infty}\left(C_{1n}\frac{\sin\mu_n\frac{r}{R}}{\frac{r}{R}}-C_{2n}\frac{\sin N\mu\frac{r}{R}}{\frac{r}{R}}\right)\times$$

$$\left.\times\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right)\right], \qquad (1.28)$$

where

$$E_{1l}=\frac{2N}{(-1)^{\alpha l}(N-K_4)\,\alpha\pi l}, \qquad E_{2l}=\frac{2K_4}{(-1)^{\beta l}(N-K_4)\,\beta\pi l},$$

$$C_{1n}=\frac{2\frac{K_4}{N}\varphi(N\mu_n)}{\psi_n}, \qquad C_{2n}=\frac{2\varphi(\mu_n)}{\psi_n},$$

$$\varphi(N\mu_n)=\sin N\mu_n-N\mu_n\cos N\mu_n,$$

$$\varphi(\mu_n)=\sin\mu_n-\mu_n\cos\mu_n,$$

$$\psi_n=\frac{K_4}{N}\mu_n\cos\mu_n\;\varphi(N\mu_n)-N\mu_n\cos N\mu_n\;\varphi(\mu_n)+$$

$$+(NK_4-1)\mu_n^2\sin\mu_n\;\sin N\mu_n.$$

The distribution of mass-transfer potential in a sphere can be obtained by a method analogous to the method used in obtaining equation (1.28).

The solutions obtained are

$$\frac{\theta_0-\theta(r,\tau)}{\theta_0}=\frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\left[\sum_{l=1}^{\infty}\left(E_{1l}^{*}\frac{\sin\alpha\pi l\frac{r}{R}}{\frac{r}{R}}-\right.\right.$$

$$\left.-E_{2l}^{*}\frac{\sin\beta\pi l\frac{r}{R}}{\frac{r}{R}}\right)\exp\left(-\frac{\beta^2\pi^2l^2}{\nu_1^2}\mathrm{Fo}\right)+$$

$$+\sum_{n=1}^{\infty}\left(C_{1n}^{*}\frac{\sin\mu_n\frac{r}{R}}{\frac{r}{R}}-C_{2n}^{*}\frac{\sin N\mu_n\frac{r}{R}}{\frac{r}{R}}\right)\exp\times$$

$$\times\left(-\frac{\mu_n^2}{\nu_1^2}\,\mathrm{Fo}\right)\Bigg], \tag{1.29}$$

where

$$E_{1l}^* = E_{1l}, \quad E_{2l}^* = \frac{1}{NK_4}\,E_{2l},$$

$$C_{1n}^* = \frac{1}{NK_4}\,C_{1n}, \quad C_{2n}^* = C_{2n}.$$

In the first term the summation is conducted with respect to the common roots of equation (1.25), and in the second with respect to the roots of the characteristic equation (1.26). α and β are integers and $N = \frac{\nu_2}{\nu_1} = \frac{\alpha}{\beta}$ is a rational fraction.

The average values of transfer potentials for a sphere are determined from

$$\bar{t}(\tau) = \frac{3}{R^3}\int_0^R r^2 t(r,\ \tau)\,dr, \qquad \bar{\theta}(\tau) = \frac{3}{R^3}\int_0^R r^2\theta(r,\ \tau)\,dr.$$

In dimensionless form the potentials are

$$\frac{t(\tau)}{t_0} = -\left[\sum_{l=1}^{\infty} F_l \exp\left[-\left(\frac{\beta\pi l}{\nu_1}\right)^2 \mathrm{Fo}\right] + \right.$$

$$\left. + \sum_{n=1}^{\infty} D_n \exp\left(-\frac{\mu_n^2}{\nu_1^2}\,\mathrm{Fo}\right)\right], \tag{1.30}$$

$$\frac{\theta_0 - \theta(\tau)}{\theta_0} = \frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\left[\sum_{l=1}^{\infty} F_l^* \exp\left[-\left(\frac{\beta\pi l}{\nu_1}\right)^2 \mathrm{Fo}\right] + \right.$$

$$\left. + \sum_{n=1}^{\infty} D_n^* \exp\left(-\frac{\mu_n^2}{\nu_1^2}\,\mathrm{Fo}\right)\right], \tag{1.31}$$

where
$$F_l = \frac{3(\sin \alpha\pi l - \alpha\pi l \cos \alpha\pi l)}{(\alpha\pi l)^2} E_{1l} -$$
$$- \frac{3(\sin \beta\pi l - \beta\pi l \cos \beta\pi l)}{(\beta\pi l)^2} E_{2l},$$
$$F_l^* = \frac{3(\sin \alpha\pi l - \alpha\pi l \cos \alpha\pi l)}{(\alpha\pi l)^2} E_{1l}^* -$$
$$- \frac{3(\sin \beta\pi l - \beta\pi l \cos \beta\pi l)}{(\beta\pi l)^2} E_{2l}^*,$$
$$D_n = \frac{3(\sin \mu_n - \mu_n \cos \mu_n)}{\mu_n^2} C_{1n} -$$
$$- \frac{3(\sin N\mu_n - N\mu_n \cos N\mu_n)}{(N\mu_n)^2} C_{2n},$$
$$D_n^* = \frac{3(\sin \mu_n - \mu_n \cos \mu_n)}{\mu_n^2} C_{1n}^* -$$
$$- \frac{3(\sin N\mu_n - N\mu_n \cos N\mu_n)}{(N\mu_n)^2} C_{2n}^*.$$

The dimensionless rates of change of the transfer potentials (local and average) are obtained by differentiation of equations (1.28) to (1.31) with respect to the dimensionless time, Fo:

$$\frac{d}{d\text{Fo}}\left[\frac{t(r,\tau)}{t_0}\right] = \sum_{l=1}^{\infty}\left(\frac{\beta\pi l}{\nu_1}\right)^2 \left(E_{1l}\frac{\sin \alpha\pi l \frac{r}{R}}{\frac{r}{R}} -\right.$$
$$\left. - E_{2l}\frac{\sin \beta\pi l \frac{r}{R}}{\frac{r}{R}}\right) \exp\left[-\left(\frac{\beta\pi l}{\nu_1}\right)^2 \text{Fo}\right] +$$
$$+ \sum_{n=1}^{\infty}\frac{\mu_n^2}{\nu_1^2}\left(C_{1n}\frac{\sin \mu_n \frac{r}{R}}{\frac{r}{R}} - C_{2n} \times\right.$$
$$\left.\times \frac{\sin N\mu_n \frac{r}{R}}{\frac{r}{R}}\right) \exp\left(-\frac{\mu_n^2}{\nu_1^2}\text{Fo}\right),$$

$$\frac{d}{d\mathrm{Fo}}\left[\frac{\bar{t}(\tau)}{t_0}\right]=\sum_{l=1}^{\infty}\left(\frac{\beta\pi l}{\nu_1}\right)^2 F_l \exp\left[-\left(\frac{\beta\pi l}{\nu_1}\right)^2 \mathrm{Fo}\right]+$$

$$+\sum_{n=1}^{\infty}\frac{\mu_n}{\nu_1^2} D_n \exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right),$$

$$\frac{d}{d\,\mathrm{Fo}}\left[\frac{\theta(r,\tau)}{\theta_0}\right]=\frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\left[\sum_{l=1}^{\infty}\left(\frac{\beta\pi l}{\nu_1}\right)^2\left(E_{1l}^{*}\frac{\sin\alpha\pi l\frac{r}{R}}{\frac{r}{R}}-\right.\right.$$

$$\left.-E_{2l}^{*}\frac{\sin\beta\pi l\frac{r}{R}}{\frac{r}{R}}\right)\exp\left[-\left(\frac{\beta\pi l}{\nu_1}\right)^2\mathrm{Fo}\right]+\sum_{n=1}^{\infty}\frac{\mu_n^2}{\nu_1^2}\times$$

$$\left.\times\left(C_{1n}^{*}\frac{\sin\mu_n\frac{r}{R}}{\frac{r}{R}}-C_{2n}^{*}\frac{\sin N\mu_n\frac{r}{R}}{\frac{r}{R}}\right)\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right)\right],$$

$$\frac{d}{d\mathrm{Fo}}\left[\frac{\bar{\theta}(\tau)}{\theta_0}\right]=\frac{1-\nu_2^2}{\varepsilon\,\mathrm{Ko}_1}\left[\sum_{l=1}^{\infty}F_l^{*}\left(\frac{\beta\pi l}{\nu_1}\right)^2\exp\times\right.$$

$$\left.\times\left[-\left(\frac{\beta\pi l}{\nu_1}\right)^2\mathrm{Fo}\right]+\sum_{n=1}^{\infty}D_n^{*}\frac{\mu_n^2}{\nu_1^2}\exp\left(-\frac{\mu_n^2}{\nu_1^2}\mathrm{Fo}\right)\right].$$

c) Analysis of the Solutions.

Equations (1.14), (1.16), (1.28) and (1.29) show that the distribution of transfer potentials is determined by the following dimensionless numbers: Fo and Ko_1 (or Ko_2), and ν_1 and ν_2. The latter, as shown in Section 1b of Chapter II, are functions of Lu and $\mathrm{Fe}=\varepsilon\,\mathrm{Ko}_1\mathrm{Pn}_1$.

Consequently,

$$\frac{t-t_0}{t_0}=f_1\left(\mathrm{Fo},\ \mathrm{Lu},\ \varepsilon\,\mathrm{Ko}_1\mathrm{Pn}_1,\frac{x}{R}\right),\qquad(1.32)$$

$$\frac{\theta_0-\theta}{\theta_0}=f_2\left(\mathrm{Fo},\ \mathrm{Ko}_1,\ \mathrm{Lu},\ \varepsilon\,\mathrm{Ko}_1\mathrm{Pn}_1,\ \frac{x}{R}\right),\qquad(1.33)$$

i.e., the heat transfer cannot be separated from mass transfer and vice versa.

By comparison of equations (1.32) and (1.33) with analogous expressions (1.32) and (1.33) of Chapter II, we can draw the following conclusions. The transition from solutions of heat and mass transfer problems with boundary conditions of the third kind to problems with boundary conditions of the first kind simplifies the analytical solutions due to the following reasons: 1) the Biot numbers are excluded from the solutions, and 2) the dimensionless numbers (ε, Ko_1, Pn_1), can be combined by means of the Fedorov number.

In conclusion we shall give an example of a calculation of the temperature and mass-transfer potential fields for an infinite plate. The heat and mass transfer properties will be assumed to have the following values:

$c_q=0{,}45$ Kcal/kg-°C; $a_q=0{,}865\cdot10^{-3}$ m²/hr; $u_0=0{,}15$ kg/kg;

$a_m=0{,}13\cdot10^{-3}$ m²/hr; $\delta=1\cdot10^{-3}$ /°C ; $t_0=90°$ C;

$\rho=560$ Kcal/kg; $\varepsilon=0{,}09$; $t_c=0°$ C.

Here we get

$$\mathrm{Lu}=\frac{a_m}{a_q}=0{,}150,$$

$$\mathrm{Ko}_1=\frac{\rho u_0}{c_q t_0}=1{,}44,$$

$$\mathrm{Fe}=\frac{\varepsilon\rho\delta}{c_q}=0{,}112,$$

$\nu_1=0{,}993$, $\nu_2=2{,}61$, $N=2{,}63$, $K_4=-147$, $\mu_1=1{,}19$, $\mu_2=1{,}58$.

Table 32. Distribution of dimensionless temperature in an infinite plate.

$\frac{x}{R}$	$\frac{t(x,\tau)}{t_0}$			
	Fo=0,1	Fo=0,5	Fo=1,0	Fo=1,5
0,0	0,900	0,290	0,096	0,024
0,2	0,918	0,338	0,098	0,029
0,4	0,783	0,287	0,083	0,024
0,6	0,568	0,208	0,060	0,017
0,8	0,295	0,107	0,030	0,009
1,0	0,000	0,000	0,000	0,000

Table 33. Distribution of dimensionless mass transfer potential in an infinite plate.

$\frac{x}{R}$	$\frac{\theta(x,\tau)-\theta_0}{\theta_0}$			
	Fo=0,1	Fo=0,5	Fo=1,0	Fo=1,5
0,0	−0,026	0,149	0,079	0,049
0,2	0,002	0,079	0,068	0,041
0,4	0,084	0,060	0,034	0,018
0,6	0,093	0,004	−0,016	−0,013
0,8	−0,041	−0,093	−0,071	−0,042
1,0	−0,275	−0,197	−0,112	−0,059

The dimensionless temperature and dimensionless mass potential, calculated from equations (1.14) and (1.16), are given as functions of $\frac{x}{R}$ in Tables 32 and 33. It should be noted that only two terms of the series were employed in the calculations.

2. HEAT AND MASS TRANSFER WITH CONSTANT MASS CONCENTRATION AND IN THE ABSENCE OF PHASE TRANSFORMATION.

The analytical description of certain cases of heat and mass transfer with boundary conditions of the first kind, (VII) and (VIII), can be obtained from the solutions of corresponding problems of the preceding section by finding limiting forms of the solutions. (See Section 5, Chapter II). The same results can be obtained from a direct solution of the incomplete ($\delta=0$, $\varepsilon=0$, etc.) system of differential equations (I) and (II). As an illustration, let us consider the solutions of the incomplete system of equations for the case in which there is no phase transformation, that is, the mass transfer in the interior of the body takes place only in the liquid phase, e.g. when within the material the temperature $t \leqslant 50°\mathrm{C}$.

a) Heat and Mass Transfer in a Sphere

$(\Gamma=2,\ \xi=r,\ \varepsilon=0, t(r,\ 0)=t_0=\mathrm{const},\ \theta(r,\ 0)=\theta_0=\mathrm{const})$.

In this case the original system of equations (I) and (II) for a sphere can be rewritten in the following form:

$$\frac{\partial\,[rt(r,\ \tau)]}{\partial\tau}=a_q\frac{\partial^2\,[rt(r,\ \tau)]}{\partial r^2}, \tag{2.1}$$

$$\frac{\partial\,[r\theta(r,\ \tau)]}{\partial\tau}=a_m\frac{\partial^2\,[r\theta(r,\ \tau)]}{\partial r^2}+a_m\delta\frac{\partial^2\,[rt(r,\ \tau)]}{\partial r^2} \tag{2.2}$$

The transform for the temperature field (2.1) is

$$t_L(r, s) - \frac{t_0}{s} = -\frac{Rt_0}{r}\,\frac{\operatorname{sh}\sqrt{\frac{s}{a_q}}\,r}{s\operatorname{sh}\sqrt{\frac{s}{a_q}}\,R}. \tag{2.3}$$

The inversion of equation (2.3) will be obtained by using the expansion theorem. If we denote

$$\Phi(s) = \operatorname{sh}\sqrt{\frac{s}{a_q}}\,r,$$

$$\psi(s) = s\operatorname{sh}\sqrt{\frac{s}{a_q}}\,R,$$

then the roots of the polynomial, ψ (s), will be

1. $s = 0$,
2. $s_n = -\frac{\mu_n^2 a_q}{R_2}$, where $\mu_n = i\sqrt{\frac{s_n}{a_q}}\,R = n\pi\ (n = 1, 2, 3\ldots)$ are

the roots of equation

$$\operatorname{sh}\sqrt{\frac{s_n}{a_q}}\,R = 0.$$

Consequently, the expansion theorem can be written in the following form:

$$\frac{t(r,\tau)}{t_0} = -\frac{Rt_0}{r}\left[\lim_{s\to 0}\frac{\Phi(s)}{\psi'(s)} + \sum_{n=1}^{\infty}\frac{\Phi(s_n)}{\psi'(s_n)}e^{s_n\tau}\right]$$

and the inversion of equation (2.3) is

$$\frac{t(r,\tau)}{t_0} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\sin n\pi\frac{r}{R}}{n\pi\frac{r}{R}}\exp(-n^2\pi^2\mathrm{Fo}). \tag{2.4}$$

For $\mathrm{Fo} > 0.1$ the series converges quite rapidly and only the first term of the series will be utilized. For $\mathrm{Fo} < 0.1$ several terms of the infinite series have to be used.

Let us apply the Laplace transformation to equation (2.3) as we did with equation (2.2):

$$a_m u_L''(r,s) - s\left[u_L(r,s) - \frac{r\theta_0}{s}\right] - \\ - \frac{R t_0 a_m \delta \,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,r}{a_q\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,R} = 0,$$

where

$$u_L(r,s) = r\theta_L(r,s).$$

The solution of this equation yields

$$\theta_L(r,s) = B_1\frac{\mathrm{ch}\sqrt{\frac{s}{a_m}}\,r}{r} + B_2\frac{\mathrm{sh}\sqrt{\frac{s}{a_m}}\,r}{r} + \frac{\theta_0}{s} + \\ + \frac{\delta t_0 R\left[\mathrm{sh}\sqrt{\frac{s}{a_q}}\,r - \sqrt{\frac{a_m}{a_q}}\,\mathrm{sh}\sqrt{\frac{s}{a_m}}\,r\right]}{r\left(1-\frac{a_q}{a_m}\right)s\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,R}. \tag{2.5}$$

The transformed condition of symmetry (V) shows that $B_1 = 0$. The coefficient B_2, which is constant with respect to r, will be determined from (VII). In order to satisfy this boundary condition we have

$$B_2 = \frac{\delta t_0 R}{(a_m - a_q)s\,\mathrm{sh}\sqrt{\frac{s}{a_q}}\,R} \times \\ \times\left[\sqrt{\frac{a_m}{a_q}}\left(\sqrt{\frac{s}{a_m}}\,R\,\mathrm{ch}\sqrt{\frac{s}{a_m}}\,R - \mathrm{sh}\sqrt{\frac{s}{a_m}}\,R\right)a_m - \\ - \left(\sqrt{\frac{s}{a_q}}\,R\,\mathrm{ch}\sqrt{\frac{s}{a_q}}\,R - \mathrm{sh}\sqrt{\frac{s}{a_q}}\,R\right)a_q\right] \times \\ \times\left(\sqrt{\frac{s}{a_m}}\,R\,\mathrm{ch}\sqrt{\frac{s}{a_m}}\,R - \mathrm{sh}\sqrt{\frac{s}{a_m}}\,R\right)^{-1}$$

Thus, the transform for the distribution of mass-transfer potential in a sphere has the following form:

$$\theta_L(r,\, s) - \frac{\theta_0}{s} = \frac{\delta\, t_0\, R}{r\,(a_m - a_q)} \times$$

$$\times \left[a_m \operatorname{sh} \sqrt{\frac{s}{a_q}}\, r \left(\sqrt{\frac{s}{a_m}}\, R \operatorname{ch} \sqrt{\frac{s}{a_m}}\, R - \operatorname{sh} \sqrt{\frac{s}{a_m}}\, R\right) -\right.$$

$$\left. - a_q \left(\sqrt{\frac{s}{a_q}}\, R \operatorname{ch} \sqrt{\frac{s}{a_q}}\, R - \operatorname{sh} \sqrt{\frac{s}{a_q}}\, R\right) \operatorname{sh} \sqrt{\frac{s}{a_m}}\, r\right] \times$$

$$\times s \operatorname{sh} \sqrt{\frac{s}{a_q}}\, R \left(\sqrt{\frac{s}{a_m}}\, R \operatorname{ch} \sqrt{\frac{s}{a_m}}\, R - \operatorname{sh} \sqrt{\frac{s}{a_m}}\, R\right)^{-1} =$$

$$= \frac{\delta\, t_0 R}{r\,(a_m - a_q)}\, \frac{\Phi(s)}{\psi(s)}\,. \tag{2.6}$$

Since the conditions of the expansion theorem are fulfilled in equation (2.6) we shall utilize this theorem to obtain the inversion of $\theta(r,\, s)$.

The roots of the polynomial $\psi(s)$ are

1) $s = 0$,

2) $s_n = -\dfrac{\mu_n^2 a_q}{R^2}$, where $\mu_n = i \sqrt{\dfrac{s_n}{a_q}}\, R = n\pi$ are roots of equation $\operatorname{sh} \sqrt{\dfrac{s}{a_q}}\, R = 0$,

3) $s_l = -\dfrac{\mu_l^2 a_m}{R^2}$, where $\mu_l = i \sqrt{\dfrac{s_l}{a_m}}\, R$ are roots of equation

$$\operatorname{tg} \mu_l = \mu_l. \tag{2.7}$$

Consequently,

$$\theta\ (r,\, s) - \theta_0 = \frac{\delta\, t_0 R}{r\,(a_m - a_q)} \left[\lim_{s \to 0} \frac{\Phi(s)}{\psi'(s)} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{\Phi(s_n)}{\psi'(s_n)}\, e^{s_n \tau} + \sum_{l=1}^{\infty} \frac{\Phi(s_l)}{\psi'(s_l)}\, e^{s_l \tau}\right]. \tag{2.8}$$

Here

$$\psi'(s) = \operatorname{sh} \sqrt{\frac{s}{a_q}}\, R \left(\sqrt{\frac{s}{a_m}}\, R \operatorname{ch} \sqrt{\frac{s}{a_m}} \cdot R - \operatorname{sh} \sqrt{\frac{s}{a_m}}\, R\right) +$$

$$+\frac{1}{2}\sqrt{\frac{s}{a_q}}R\operatorname{ch}\sqrt{\frac{s}{a_q}}R\left(\sqrt{\frac{s}{a_m}}R\operatorname{ch}\sqrt{\frac{s}{a_m}}R-\operatorname{sh}\sqrt{\frac{s}{a_m}}R\right)+$$

$$+\frac{1}{2}\sqrt{\frac{s}{a_m}}R\sqrt{\frac{s}{a_m}}R\operatorname{sh}\sqrt{\frac{s}{a_m}}R\operatorname{sh}\sqrt{\frac{s}{a_q}}R.$$

In order to obtain the first term of equation (2.8) we shall expand the numerator and denominator, $\frac{\Phi(s)}{\psi(s)}$, in a series; divide the fraction by $\sqrt{\frac{s}{a_m}}\times\times\sqrt{\frac{s}{a_q}}$, differentiate the denominator with respect to s, and take the limit of this fraction as $s\to 0$. These operations yield

$$\lim_{s\to 0}\frac{\Phi(s)}{\psi'(s)}=0.$$

For the remaining terms, we get

$$\Phi(s_n)=a_m\sin\mu_n\frac{r}{R}\left(\sin\sqrt{\frac{a_q}{a_m}}\mu_n-\sqrt{\frac{a_q}{a_m}}\mu_n\cos\sqrt{\frac{a_q}{a_m}}\mu_n\right)+$$

$$+a_q\mu_n\cos\mu_n\sin\sqrt{\frac{a_q}{a_m}}\mu_n\frac{r}{R},$$

$$\Phi(s_l)=-\left(\sin\sqrt{\frac{a_m}{a_q}}\mu_l-\sqrt{\frac{a_m}{a_q}}\mu_l\cos\sqrt{\frac{a_m}{a_q}}\mu_l\right)a_q\sin\mu_l\frac{r}{R},$$

$$\psi'(s_n)=-\frac{1}{2}\mu_n\cos\mu_n\left(\sqrt{\frac{a_q}{a_m}}\mu_n\cos\sqrt{\frac{a_q}{a_m}}\mu_n-\right.$$

$$\left.-\sin\sqrt{\frac{a_q}{a_m}}\mu_n\right),$$

$$\psi'(s_l)=\frac{1}{2}\mu_l^2\sin\mu_l\sin\sqrt{\frac{a_m}{a_q}}\mu_l.$$

Consequently, the expression for the distribution of mass-transfer potential in a sphere can be written as

$$\frac{\theta(r,\tau)-\theta_0}{\theta_0}=\frac{2\mathrm{LuPn}_1}{\mathrm{Lu}-1}\times$$

$$\times\left\{\sum_{n=1}^{\infty}\frac{\exp(-n^2\pi^2\mathrm{Fo})}{(-1)^n n\pi\left(\sin\frac{n\pi}{\sqrt{\mathrm{Lu}}}-\frac{n\pi}{\sqrt{\mathrm{Lu}}}\cos\frac{n\pi}{\sqrt{\mathrm{Lu}}}\right)}\times\right.$$

$$\times\left[\left(\sin\frac{n\pi}{\sqrt{\mathrm{Lu}}}-\frac{n\pi}{\sqrt{\mathrm{Lu}}}\cos\frac{n\pi}{\sqrt{\mathrm{Lu}}}\right)\frac{\sin n\pi\frac{r}{R}}{\frac{r}{R}}+\right.$$

$$\left.+(-1)^n\frac{n\pi}{\sqrt{\mathrm{Lu}}}\;\frac{\sin\frac{n\pi}{\sqrt{\mathrm{Lu}}}\,\frac{r}{R}}{\frac{r}{R}}\right]-$$

$$-\sum_{l=1}^{\infty}\frac{(\sin\sqrt{\mathrm{Lu}}\,\mu_l-\sqrt{\mathrm{Lu}}\,\mu_l\cos\sqrt{\mathrm{Lu}}\,\mu_l)}{\mathrm{Lu}\,\mu_l^2\sin\mu_l\,\sin\sqrt{\mathrm{Lu}}\,\mu_l}\times$$

$$\left.\times\frac{\sin\mu_l\frac{r}{R}}{\frac{r}{R}}\exp(-\mu_l^2\,\mathrm{LuFo})\right\}, \qquad (2.9)$$

where μ_l are the roots of the characteristic equation (2.7). The values of the roots are given in Table 34.

Table 34.
Roots μ_l of equation $\mathrm{tg}\,\mu=\mu$

l	μ_l
1	0,0000
2	4,4934
3	7,7253
4	10,9041
5	14,0662
6	17,2208

b) Heat and Mass Transfer in an Infinite Cylinder

$(\Gamma=1, \xi=r,\ \varepsilon=0,\ t(r,0)=t_0=\mathrm{const},\ \theta(r,0)=\theta_0=\mathrm{const})$.

The relative temperature and dimensionless mass-transfer potential at any point in an infinite cylinder will have the following forms, respectively:

$$\frac{t(r,\tau)}{t_0}=\sum_{n=1}^{\infty}\frac{2}{\mu_n J_1(\mu_n)}J_0\left(\mu_n\frac{r}{R}\right)\exp(-\mu_n^2\,\mathrm{Fo}), \qquad (2.10)$$

$$\begin{aligned}\frac{\theta(r,\tau)-\theta_0}{\theta_0} &= \frac{2\mathrm{LuPn}_1}{\mathrm{Lu}-1}\left\{\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{\mathrm{Lu}}}\times\right.\right.\\ &\left.\times\frac{J_0\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\frac{r}{R}\right)}{\mu_n J_1\left(\frac{\mu_n}{\sqrt{\mathrm{Lu}}}\right)}-\frac{J_0\left(\mu_n\frac{r}{R}\right)}{\mu_n J_1(\mu_n)}\right]\exp\left(-\mu_n^2\,\mathrm{Fo}\right)-\\ &-\sum_{l=1}^{\infty}\frac{J_1\left(\sqrt{\mathrm{Lu}}\,\mu_l\right)}{\sqrt{\mathrm{Lu}}\,\mu_l J_0\left(\sqrt{\mathrm{Lu}}\,\mu_l\right)J_0'(\mu_l)}\,J_0\left(\mu_l\frac{r}{R}\right)\times\\ &\left.\times\exp\left(-\mu_l^2\,\mathrm{LuFo}\right)\right\}, \qquad (2.11)\end{aligned}$$

where μ_n are the roots of the characteristic equation

$$J_0(\mu)=0,$$

and μ_l are the roots of the characteristic equation

$$J_1(\mu)=0.$$

The values of the characteristic roots μ_n and μ_l are given in Table 35.

Table 35

Roots of the characteristic equations

$J_0(\mu)=0$ and $J_1(\mu)=0$

n or l	Roots μ_n of equation $J_0(\mu)=0$.	Roots μ_l of equation $J_1(\mu)=0$
1	2,4048	3,8317
2	5,5201	7,0156
3	8,6537	10,1735
4	11,7915	13,3237
5	14,9309	16,4706
6	18,0711	19,6159
7	21,2116	22,7601
8	24,3525	25,9037
9	27,4935	29,0468
10	30,6346	32,1897

c) Heat and Mass Transfer in an Infinite Plate
($\Gamma=0$, $\xi=x$, $\varepsilon=0$, $t(x,\ 0)=t_0=\text{const}$, $\theta(x,\ 0)=\theta_0=\text{const}$).

Here, the distribution of heat and mass-transfer potentials have the following form:

$$\frac{t(x,\tau)}{t_0}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2\cos(2n-1)\frac{\pi}{2}\frac{x}{R}}{(2n-1)\frac{\pi}{2}}\times$$

$$\times\exp\left[-\frac{(2n-1)^2\pi^2}{4}\,\text{Fo}\right], \tag{2.12}$$

$$\frac{\theta(x,\tau)-\theta_0}{\theta_0}=\frac{2\,\text{Lu}\,\text{Pn}_1}{\text{Lu}-1}\times$$

$$\times\left\{\sum_{n=1}^{\infty}\frac{\exp\left[-\frac{(2n-1)^2\pi^2}{4}\,\text{Fo}\right]}{(-1)^{n+1}(2n-1)\frac{\pi}{2}\sin\frac{(2n-1)\frac{\pi}{2}}{\sqrt{\text{Lu}}}}\times\right.$$

$$\times\left[(-1)^{n+1}\frac{1}{\sqrt{\text{Lu}}}\cos\frac{(2n-1)\frac{\pi}{2}\frac{x}{R}}{\sqrt{\text{Lu}}}-\right.$$

$$\left.-\sin\frac{(2n-1)\frac{\pi}{2}}{\sqrt{\text{Lu}}}\cos(2n-1)\frac{\pi}{2}\frac{x}{R}\right]-$$

$$-\sum_{l=1}^{\infty}\frac{\frac{1}{\sqrt{\text{Lu}}}\sin\sqrt{\text{Lu}}\,\pi l}{(-1)^l\pi l\cos\sqrt{\text{Lu}}\,\pi l}\times$$

$$\left.\times\cos\pi l\frac{x}{R}\ \exp(-\pi^2l^2\,\text{Lu}\,\text{Fo})\right\}, \tag{2.13}$$

where $n=1, 2, 3..$, $l=1, 2, 3....$

d) Heat and Mass Transfer in a Finite Rod
($\Gamma=0$, $\xi=x$, $\varepsilon=0$, $t(x,0)=t_0=\text{const}$, $\theta(x,0)=\theta_0=\text{const}$).

Let us consider that one end of the rod, located in the origin of the coordinates ($x=0$), is at a constant temperature during the entire experiment. This temperature is equal to the temperature of the external medium (heat carrier), $t_{\text{н}}$,

$$t(0,\tau)=t_{\text{н}}=\text{const}. \tag{2.14}$$

The other end of the rod $(x=l)$ remains at the (initial) temperature of the medium during the experiment, i.e.,

$$t(l,\tau)=t_c=t_0=\text{const}, \tag{2.15}$$

and $t_{\text{н}}>t_0$. The side surfaces of the rod are insulated against heat and mass transfer.

The general solution of the system of equations (I) and (II) with uniform initial conditions (VI) and boundary conditions (VII), (2.14) and (2.15), has the following form:

$$\frac{t(x,\tau)-t_0}{t_{\text{н}}-t_0}=\left(1-\frac{x}{l}\right)+ \\ +\sum_{n=1}^{\infty}(-1)^n\frac{2}{\pi n}\sin\pi n\left(1-\frac{x}{l}\right)\exp\left(-\pi^2n^2\text{Fo}_1\right), \tag{2.16}$$

$$\frac{\theta(x,\tau)-\theta_0}{\theta_0}=\text{Pn}_2\left(\frac{x}{l}-\frac{1}{2}\right)+\frac{2\,\text{Lu}\,\text{Pn}_2}{\text{Lu}-1}\times \\ \times\left\{\sum_{n=1}^{\infty}\frac{\exp\left(-n^2\pi^2\text{Fo}_1\right)}{n\pi\sin\frac{n\pi}{\sqrt{\text{Lu}}}}\left[\frac{(-1)^n}{\sqrt{\text{Lu}}}\cos\frac{n\pi}{\sqrt{\text{Lu}}}\frac{x}{l}+\right.\right. \\ +\sin\frac{n\pi}{\sqrt{\text{Lu}}}\sin n\pi\frac{x}{l}-\frac{1}{\sqrt{\text{Lu}}}\times \\ \left.\times\cos\left(1-\frac{x}{l}\right)\frac{n\pi}{\sqrt{\text{Lu}}}\right]+\sum_{k=1}^{\infty}\frac{(-1)^k-\cos\sqrt{\text{Lu}}\,\pi k}{\sqrt{\text{Lu}}\,\pi k\,\sin\sqrt{\text{Lu}}\,\pi k}\times \\ \left.\times\cos k\pi\frac{x}{l}\exp\left(-k^2\pi^2\,\text{Lu}\,\text{Fo}_1\right)\right\}. \tag{2.17}$$

In the above, $n=1, 2, 3\ldots$, $k=1, 2, 3\ldots$,

$$\text{Pn}_2=\frac{\delta(t_{\text{н}}-t_0)}{\theta_0}\ \text{and}\ \text{Fo}_1=\frac{a_q\tau}{l^2}.$$

Equations (2.16) and (2.17) can conveniently be used for large values of Fo_1, i.e., for $\text{Fo}_1>0.1$, for then only two terms of the infinite series need be used for calculations. At low values of Fo_1, accurate results can be obtained only when many terms of the series are taken into account and therefore it is convenient to utilize much more rapidly converging expressions:

$$\frac{t(x,\tau)-t_0}{t_{\text{н}}-t_0} = \operatorname{erfc}\frac{\frac{x}{l}}{2\sqrt{\text{Fo}_1}} +$$

$$+\sum_{n=1}^{\infty}\left\{\operatorname{erfc}\frac{2n+\frac{x}{l}}{2\sqrt{\text{Fo}_1}} - \operatorname{erfc}\frac{2n-\frac{x}{l}}{2\sqrt{\text{Fo}_1}}\right\}, \tag{2.18}$$

$$\frac{\theta(x,\tau)-\theta_0}{\theta_0} = \frac{\text{Lu}\,\text{Pn}_2}{\text{Lu}-1}\left\{\frac{1}{\sqrt{\text{Lu}}}\operatorname{erfc}\frac{\frac{x}{l}}{2\sqrt{\text{Lu}\,\text{Fo}_1}} +\right.$$

$$+\frac{1}{\sqrt{\text{Lu}}}\sum_{k=1}^{\infty}\left[\operatorname{erfc}\frac{2k+\frac{x}{l}}{2\sqrt{\text{LuFo}_1}} + \operatorname{erfc}\frac{2k-\frac{x}{l}}{2\sqrt{\text{Lu}\,\text{Fo}_1}}\right] -$$

$$-\operatorname{erfc}\frac{\frac{x}{l}}{2\sqrt{\text{Fo}_1}} - \sum_{n=1}^{\infty}\left[\operatorname{erfc}\frac{2n+\frac{x}{l}}{2\sqrt{\text{Fo}_1}} - \operatorname{erfc}\frac{2n-\frac{x}{l}}{2\sqrt{\text{Fo}_1}}\right] +$$

$$+\frac{2}{\sqrt{\text{Lu}}}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^n\left[\operatorname{erfc}\frac{n+\frac{k+\frac{x}{l}}{\sqrt{\text{Lu}}}}{2\sqrt{\text{Fo}_1}} +\right.$$

$$\left.+\operatorname{erfc}\frac{n-\frac{k-\frac{x}{l}}{\sqrt{\text{Lu}}}}{2\sqrt{\text{Fo}_1}}\right] +$$

$$\left.+\frac{2}{\sqrt{\text{Lu}}}\sum_{n=1}^{\infty}\operatorname{erfc}\frac{2n+\frac{\frac{x}{l}}{\sqrt{\text{Lu}}}}{2\sqrt{\text{Fo}}}\right\}. \tag{2.19}$$

e) Analysis of Solutions.

The temperature field is analogous to a temperature field in a "dry" material. The relative temperature of the body is a function of the heat-transfer Fourier number and of the relative coordinate $\frac{\xi}{R}$:

$$\frac{t(\xi,\tau)}{t_0} = f\left(\text{Fo},\ \frac{\xi}{R}\right).$$

Since the characteristic roots increase with the index, beginning with $\mathrm{Fo} > \mathrm{Fo}_1$, we can neglect all terms of the series with the exception of the first. In this case the relation between the dimensionless temperature and time can be written in the following form:

$$\frac{t(\xi, \tau)}{t_0} = A_1 \Phi\left(\mu_1, \frac{\xi}{R}\right) \exp\left(-\mu_1^2 \mathrm{Fo}\right).$$

Here A_1 is an initial thermal amplitude; for an infinite plate $A_1 = \frac{4}{\pi}$, for a sphere $A_1 = \frac{2}{\pi}$, and for an infinite cylinder

$$A_1 = \frac{2}{2{,}405\ I_1\ (2{,}405)};$$

$\Phi\left(\mu_1, \frac{\xi}{R}\right)$ is a function describing the temperature distribution along a characteristic coordinate $\frac{\xi}{R}$,

μ_1 is the first root of the characteristic equation (for an infinite plate $\mu_1 = \frac{\pi}{2}$, for a sphere $\mu_1 = \pi$, for an infinite cylinder $\mu_1 = 2{,}405$);

I_1 is the modified Bessel function of the first kind and of the first order.

In a stationary state, i.e., at $\mathrm{Fo} \to \infty$, the dimensionless temperatures for plate, cylinder and sphere become equal to zero and for a finite rod $\frac{t(x, \tau) - t_0}{t_H - t_0} = 1 - \frac{x}{l}$, i.e., a linear function of x/l.

The field of the dimensionless mass-transfer potential must be described not only by Fo and $\frac{x}{R}$, but also by the Lu and Pn numbers. Analytical solutions for $\frac{\theta(\xi, \tau) - \theta_0}{\theta_0}$ represent a difference of two series, of which the first describes the modified temperature field and the second describes the modified mass potential field.

As the dimensionless time increases, the distribution of mass transfer potential tends to a stationary state, during which for the infinite plate, cylinder and a sphere $\theta(\xi, \infty) = \theta_0$, and for the finite rod:

$$\frac{\theta(x, \tau) - \theta_0}{\theta_0} = \mathrm{Pn}_2\left(\frac{x}{l} - \frac{1}{2}\right).$$

The values of $\Delta = \frac{\theta(x, \tau) - \theta_0}{\theta_0 \mathrm{Pn}_2}$ calculated by E. I. Sizyakova for a finite rod are given in Figure 41 as a function of Fo, x/l and Lu. The values of $\frac{\theta(x, \tau) - \theta_0}{\theta_0}$ were calculated from equation (2.17). The curves show that, under the influence of a variable temperature field in a moist material, a wave of mass-transfer potential is established. This wave is extinguished with time. Calculations show that at low values of Lu the wave is extinguished very rapidly. With increasing values of Lu this wave penetrates the material very deeply, is propagated at a rapid rate, and the value of its maximum increases.

The rate of change of the mass-transfer potential at every point increases considerably with increasing Lu, and the linear stationary distribution of the potential is reached very rapidly. For example, at Lu=0.8 and Fo= 0.5 the distribution of moisture in a finite rod can be considered stationary. The local temperature (Fig. 42) increases rapidly at first and then slowly approaches a constant value which corresponds to the stationary linear temperature distribution in the sample. A comparison of curves depicting the change of dimensionless transfer potentials at the same time shows that the most intensive change of mass-transfer potential takes place in the period of the most intensive change of the heat transfer potential. At low values of Lu, the change of θ takes place much more slowly than the changes of t. The stationary distribution of temperature is reached much earlier than the stationary distribution of the mass transfer potential. Calculations also show that the curves depicting the distribution of θ with respect to time show a maximum,

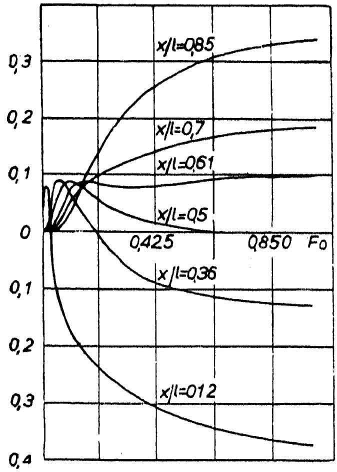


Fig. 41. $\Delta = \frac{\theta(x, \tau) - \theta_0}{\theta_0 \mathrm{Pn}_2}$ as a function of the Fourier number and of the dimensionless coordinate (Lu = 0.355).

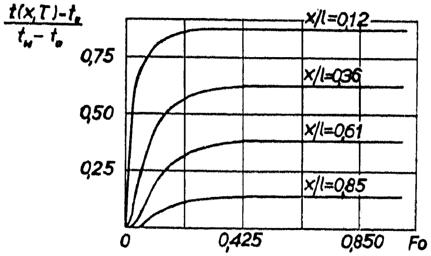


Fig. 42. Dimensionless temperature as a function of time.

and the value of the maximum at a given value of Lu is the same for all coordinate values, but increases with increasing Lu.

The dependence of heat and mass transfer processes on the Lu number with boundary conditions of the first kind is in complete accord with the analysis of the dependence of heat and mass transfer as a function of Lu with boundary conditions of the third kind (See Chapter II, Section 6). An experimental check of conclusions reached in the analytical solution of the incomplete system ($\varepsilon = 0$), was conducted by E. I. Sizyakova, who showed the presence of the maxima. However, in comparison with the theoretical values these maxima were not as pronounced and were reached much later than predicted by theory. Similarly, equilibrium was reached much later than predicted by the analytical solutions. A comparison of the analytical and experimental temperature curves shows that the actual increase of temperature takes place much more smoothly, and that the stationary distribution is reached much later than predicted. These results tend to show that there is phase transformation in the material (vaporization). Consequently, we can reach again the conclusion that it is necessary to depict the process of heat and mass transfer by a complete system of differential equations (I) and (II).

3. METHODS FOR DETERMINING HEAT AND MASS TRANSFER COEFFICIENTS. (TRANSPORT PROPERTIES FOR HEAT AND MASS TRANSFER)

S. P. Kuznetsov and E. I. Sizyakova, using the solutions for non-stationary potential fields of heat- and mass-transfer potentials given in Section 2, have proposed methods for determining heat- and mass-transfer coefficients (transport properties for heat and mass transfer). The practical application of these methods is limited because the absence of phase transformation was assumed, i.e. mass transfer was considered to occur only in the liquid phase. However, this first attempt at using the analytical solutions of the system of equations (I) and (II) can serve as a beginning of the development of new methods for determining transfer coefficients in transient experiments.

a) The Kuznetsov Method.

S. P. Kuznetsov used a spherical sample. His method permits the determination of the potential conductivity coefficient, a_m, and of the thermal gradient coefficient, δ, from one experiment. The value of a_q is found by another, independent method. The determination of the coefficients requires knowledge of the temperature and mass-transfer potential distributions in the material.

S. P. Kuznetsov used the following results of the analytical solutions as the basis of his method:

1 — the dependence of $\Delta = \frac{u(r, \tau) - u_0}{u_0 \mathrm{Pn}_1}$ on the Fo number. This dependence, as shown in Figure 43, is a unique function of Fo from 0.09 to 0.5, and of Lu from 0 to 0.5;

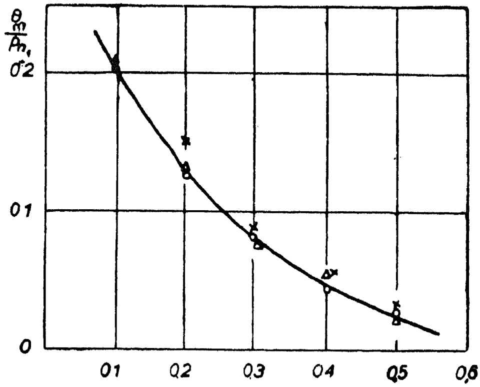


Fig. 43. $\frac{\bar{\theta}_n}{\mathrm{Pn}_1}$ as a function of the Fourier number with: $0.09 < \mathrm{Fo} < 0{,}5$ and $0 < \mathrm{Lu} < 0{,}5$: * — Lu = 0,2, 0 — Lu = 0,3, Δ — Lu = 0,4.

2 — at large values of Fo the mass-transfer potential distribution curve intersects the straight line of the initial distribution at a certain point $\left(\frac{x}{R}\right)_A$ (Fig. 44).

Figure 45 shows that this point is a unique function of the Lu number for the Fo numbers from 0.09–0.30.

We shall now illustrate the method with an example of the calculation of heat and mass transfer coefficients for sand with the following initial transfer conditions: $u_0 = 10.9\%$, and $t = 54°$ C. The radius of the flask which contained the sand was 10 cm. The coefficient of thermal diffusivity of the sand was $a_q = 2.3 \; 10^{-3}$ m²/hr. Let $\tau = 25$ min. Then, the Fourier number will be

$$\mathrm{Fo} = \frac{a_q \tau}{R^2} = \frac{2{,}2 \cdot 10^{-3} \cdot 25}{1 \cdot 10^{-2} \cdot 60} = 0{,}096.$$

Figure 43 shows that at Fo = 0.096 the value of

$$\Delta_R = \frac{u(R, \tau) - u_0}{u_0 \mathrm{Pn}_1} = 0{,}212.$$

The value of the dimensionless mass-transfer potential on the surface $\frac{u(R,\tau)-u_0}{u_0}$ is found from the distribution curves given in Figure 44:

$$\frac{u(R,\tau)-u_0}{u_0}=\frac{13{,}3-10{,}9}{10{,}9}=0{,}22.$$

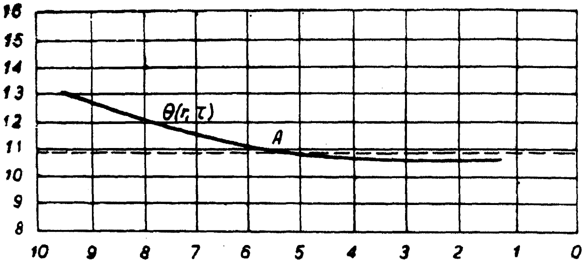


Fig. 44. Distribution of moisture content of the sand with radius, r (cm), during cooling.

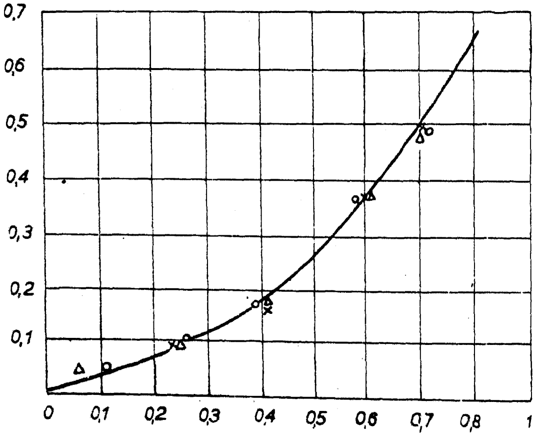


Fig. 45. Dependence of the Lu number on the relative coordinate ($\frac{r}{R}$) corresponding to the intersection of the moisture distribution curve, in the cooling process, with the lines of initial moisture (notation the same as in Figure 43).

The Posnov number is

$$\mathrm{Pn}_1 = \frac{u(R,\tau) - u_0}{u_0 \Delta_R} = \frac{0{,}22}{0{,}212} = 1{,}03.$$

Consequently, the thermal gradient coefficient is

$$\delta = \frac{u_0 \mathrm{Pn}_1}{t_0} = \frac{0{,}109 \cdot 1{,}03}{54} = 0{,}207 \cdot 10^{-2}\ 1/\,^{\circ}\mathrm{C} = 0{,}207\ \%/^{\circ}\mathrm{C}.$$

Figure 44 is used to determine the coordinate $\left(\frac{r}{R}\right)_{\mathrm{A}}$ at which the curve $u(r,\tau)$ intersects the line u_0. This intersection is at 0.5. Figure 45 is used to find that this number corresponds to Lu = 0.26. Then the coefficient

$$a_m = \mathrm{Lu}\ a_q = 0{,}26 \cdot 2{,}3 \cdot 10^{-3} = 0{,}6 \cdot 10^{-3}\ \mathrm{m^2/hr}.$$

b) The Sizyakova Method.

In accordance with the analytical solutions the distribution of mass-transfer potential with respect to time and with respect to position coordinates at low values of Fo exhibits a maximum. The magnitude of the maximum at a given value of Lu is the same for all position coordinates; however, with increasing values of Lu the maximum also increases. This behavior of the mass-transfer potential field led E. I. Sizyakova to propose a new, rapid method for determining heat and mass transfer coefficients from one experiment. For this purpose it is necessary to construct distribution curves of mass-transfer potential and temperature within a rod during a small time interval (about one hour) from the beginning of the experiment. The distribution curve for the mass-transfer potential or for the moisture content exhibits a maximum, θ_{max}, at a certain point, $\left(\frac{x}{l}\right)_{max}$. For the same point and time, the temperature of the sample is $t\left[\left(\frac{x}{l}\right)_{max}, \tau_{max}\right]$. Knowing the latter, $t_{\text{н}}$ and t_0 and using equation (2.18) we can find the corresponding values of the Fourier number, which are denoted as Fo_{max}. The latter operation is easily performed because at low values of Fo it is sufficient to use only the first term, i.e.,

$$\frac{t\left[\left(\frac{x}{l}\right)_{max}, \tau_{max}\right] - t_0}{t_{\text{н}} - t_0} = \operatorname{erfc} \frac{\frac{x}{l}}{2\sqrt{\mathrm{Fo}_1}}. \qquad (3.1)$$

Substituting into the left hand part of equation (3.1) the known values and using tables for the values of the complementary error function we find Fo_{max}. Thus, the coefficient of thermal diffusivity is

$$a_q = \frac{\mathrm{Fo}_{max}\, l^2}{\tau_{max}}.$$

Calculations show that the dependence of ln Fo_{max} on ln Lu for every value of the coordinate $\frac{x}{l}$ is linear. Knowing Fo_{max} and the value of the coordinate $\left(\frac{x}{l}\right)_{max}$, from Figure 46 we can find the value of ln Lu, and then the value of Lu. The coefficient of potential conductivity is

$$a_m = a_q\,\mathrm{Lu}.$$

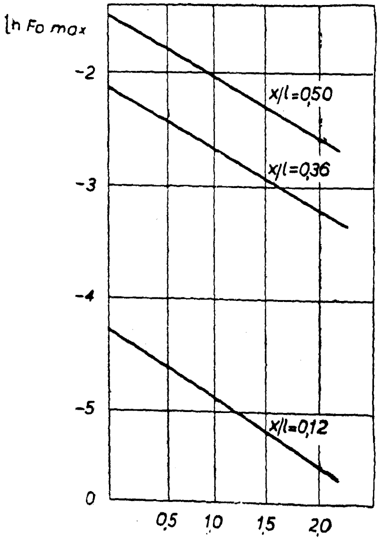


Fig. 46. Fo_{max} as a function of ln Lu.

In analyzing the dependence of Δ it was shown that Δ_{max} is neither a function of the position coordinate nor of the Fo number, but depends only on the Lu number. This permits us to find the magnitude of the Pnl number, and, consequently, of the thermal gradient coefficient, δ. Thus, from Figure 47 and on the basis of previously determined value of Lu we find Δ_{max}. Hence we can determine the Posnov number:

$$\mathrm{Pn}_1 = \frac{\theta_{max} - \theta_0}{\theta_0\,\Delta_{max}}.$$

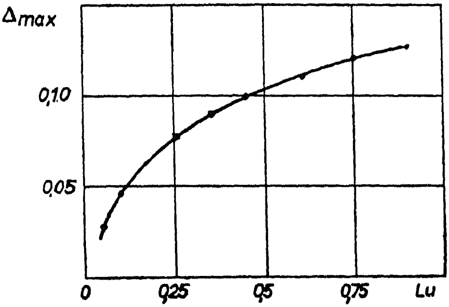


Fig. 47. $\Delta_{\max}$ as a function of Lu.

Then the thermal gradient coefficient is

$$\delta = \frac{c_m \mathrm{Pn}_1 \theta_0}{t_{\text{н}} - t_0} = \frac{\mathrm{Pn}_1\, u_0}{t_{\text{н}} - t_0}.$$

The method of E. I. Sizyakova permits the determination of the following coefficients from only one experiment: a_m, a_q and δ. In this regard the method is more convenient than the Kuznetsov method; however, as with the Kuznetsov method, it is limited to low temperatures of the material ($\leqslant 50°\mathrm{C}$) and to a narrow range of Fo values. When the Kuznetsov and Sizyakova methods are not properly applicable, the transient methods developed in the next chapter should be used.

CHAPTER IV

Non-Stationary Fields of Heat and Mass Transfer Potentials with Boundary Conditions of the Second Kind

In many cases the heat and mass fluxes at the surface of a body cannot be described by convective heat and mass transfer equations. In such circumstances the heat flux, $q_q(\tau)$, and mass flux $q_m(\tau)$ are considered as certain unknown functions of time which can be determined from experiments.

When this is the case, the boundary conditions for uniform bodies with symmetrical fields of heat and mass transfer potentials can be written:

$$-\lambda_q \frac{\partial t(R,\tau)}{\partial \xi} + q_q = 0, \qquad \text{(IX)}$$

$$-\lambda_m \frac{\partial \theta(R,\tau)}{\partial \xi} + \lambda_m \delta \frac{\partial t(R,\tau)}{\partial \xi} + q_m = 0, \qquad \text{(X)}$$

$$\frac{\partial t(0,\tau)}{\partial \xi} = 0, \quad \frac{\partial \theta(0,\tau)}{\partial \xi} = 0. \qquad \text{(XI)}$$

It should be noted that in physical significance the above boundary conditions encompass a wide variety of heat and mass transfer phenomena, including heat transfer by radiation. Initially let us consider problems in which the coefficient $\delta = 0$. Solutions for infinite plates, cylinders and spheres were obtained by M. S. Smirnov by means of finite Fourier transformations and Hankel transformations. The following system of differential transfer equations was solved:

$$\frac{\partial t(\xi,\tau)}{\partial \tau} = a_q\left[\frac{\partial^2 t(\xi,\tau)}{\partial \xi^2} + \frac{\Gamma}{\xi}\frac{\partial t(\xi,\tau)}{\partial \xi}\right] + \\ + \frac{c'_T \varepsilon \rho}{c_q}\frac{\partial \theta(\xi,\tau)}{\partial \tau}, \qquad \text{(XII)}$$

$$\frac{\partial\theta(\xi,\tau)}{\partial\tau} = a_m\left[\frac{\partial^2\theta(\xi,\tau)}{\partial\xi^2} + \frac{\Gamma}{\xi}\,\frac{\partial\theta(\xi,\tau)}{\partial\xi}\right], \qquad \text{(XIII)}$$

$$t(\xi,0) = f(\xi),\quad \theta(\xi,0) = \varphi(\xi), \qquad \text{(XIV)}$$

$$\left.\begin{aligned} &\frac{\partial t(0,\tau)}{\partial\xi} = 0,\quad t(0,\tau) < \infty \\ &\frac{\partial\theta(0,\tau)}{\partial\xi} = 0,\quad \theta(0,\tau) < \infty \end{aligned}\right\}, \qquad \text{(XV)}$$

$$-\lambda_q\frac{\partial t(R,\tau)}{\partial\xi} + q_q = 0, \qquad \text{(XVI)}$$

$$\lambda_m\frac{\partial\theta(R,\tau)}{\partial\xi} + q_m = 0. \qquad \text{(XVII)}$$

1. INFINITE PLATE ($\delta = 0$)

$$[\Gamma = 0,\ \xi = x,\ \delta = 0,\ t(x,0) = f(x),\ \theta(x,0) = \varphi(x),$$
$$q_q = \text{const},\quad q_m = \text{const}]$$

The solution of the system of differential equations (XII)-(XIII) with boundary conditions (XVI) and (XVII), initial distribution of transfer potentials (XIV), conditions of symmetry (XV) and heat and mass fluxes of constant strength will be found by means of finite Fourier transformations.

In the first place we shall determine the distribution of mass-transfer potential in an infinite plate. For this purpose equation (XIII) will be solved.

Applying the integral transformation

$$\theta_{c.F}(n,\tau) = \int_0^R \theta(x,\tau)\cos n\pi\frac{x}{R}\,dx \quad (n = 0, 1, 2, 3 \ldots) \qquad \text{(1.I)}$$

to $\theta''(x,\tau)$, we get

$$\int_0^R \frac{\partial^2\theta(x,\tau)}{\partial x}\cos n\pi\frac{x}{R}\,dx = (-1)^n\frac{\partial\theta(R,\tau)}{\partial x} -$$
$$-\frac{\partial\theta(0,\tau)}{\partial x} - \frac{n^2\pi^2}{R^2}\theta_{c.F}(n,\tau),$$

and making use of conditions (XV) and (XVII), we get

$$\int_0^R \frac{\partial^2 \theta(x,\tau)}{\partial x} \cos n\pi \frac{x}{R}\, dx =$$
$$= -(-1)^n \frac{q_m}{\lambda_m} - \frac{n^2\pi^2}{R^2} \theta_{c.F}(n,\tau). \tag{1. 2}$$

Multiplying both sides of equation (XIII) by $\cos \frac{n\pi x}{R}$, integrating between the limits 0 and R, and considering the results of equation (1.2), we get an ordinary differential equation with respect to τ:

$$\frac{d\theta_{c.F}(n,\tau)}{d\tau} + \frac{a_m n^2 \pi^2}{R^2} \theta_{c.F}(n,\tau) = -(-1)^n \frac{q_m}{\gamma_0 c'_T}.$$

Solution of this equation will be as follows:

$$\theta_{c.F}(n,\tau) = (-1)^n \frac{q_m}{\lambda_m} \frac{R^2}{n^2\pi^2} \left[\exp\left(-n^2\pi^2 \frac{a_m \tau}{R^2} \right) - 1 \right] +$$
$$+ C_1(n) \exp\left(-n^2\pi^2 \frac{a_m\tau}{R^2} \right). \tag{1. 3}$$

The integration constant, $C_1(n)$ is found from the initial condition $\theta(x,0) = \varphi(x)$. From (1.3) we get as $\tau \to 0$

$$\theta_{c.F}(n,0) = C_1(n),$$

However, in accordance with equation (1.1) and considering the initial distribution of mass-transfer potentials, we get

$$\theta_{c.F}(n,0) = \int_0^R \varphi(x) \cos n\pi \frac{x}{R}\, dx.$$

Consequently,

$$C_1(n) = \int_0^R \varphi(x) \cos n\pi \frac{x}{R}\, dx,$$

and finally the solution for the transform can be written as

$$\theta_{c.F}(n,\tau) = (-1)^n \frac{q_m}{\lambda_m} \frac{R^2}{n^2\pi^2} \left[\exp\left(-n^2\pi^2 \frac{a_m\tau}{R^2} \right) - 1 \right] +$$
$$+ \exp\left(-n^2\pi^2 \frac{a_m\tau}{R^2} \right) \int_0^R \varphi(x) \cos n\pi \frac{x}{R}\, dx. \tag{1. 4}$$

In order to obtain the distribution of mass-transfer potential we have to obtain the inversion, $\theta(x, \tau)$, of the transform $\theta_{c.F}(n, \tau)$. This can be accomplished by using the Fourier cosine transformation formula:

$$\theta(x, \tau) = \frac{1}{R}\theta_{c.F}(0, \tau) + \frac{2}{R}\sum_{n=1}^{\infty}\theta_{c.F}(n, \tau)\cos n\pi\frac{x}{R}. \tag{1.5}$$

For $n = 0$ the first term of equation (1.4) represents an indeterminacy of the kind $\frac{0}{0}$, which can be solved using L'Hopital's rule. Performing the necessary transformation for the zero term of equation (1.5), we get

$$\frac{1}{R}\theta_{c.F}(0, \tau) = -\frac{q_m\tau}{c_T\gamma_0 R} + \frac{1}{R}\int_0^R \varphi(x)\,dx. \tag{1.6}$$

Substituting equations (1.4) and (1.6) into equation (1.5) and considering that the Fourier series

$$2R\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2\pi^2}\cos n\pi\frac{x}{R} = -\frac{R}{6}\left(1 - 3\frac{x^2}{R^2}\right),$$

the distribution $\theta(x, \tau)$ in an infinite plate is found to be

$$\theta(x, \tau) - \frac{1}{R}\int_0^R \varphi(x)\,dx = -$$

$$-\frac{q_m R}{\lambda_m}\left[\mathrm{Fo}^* - \frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right) + \right.$$

$$+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n^2\pi^2}\cos n\pi\frac{x}{R}\exp(-n^2\pi^2\mathrm{Fo}^*) +$$

$$+\frac{2}{R^2}\sum_{n=1}^{\infty}\cos n\pi\frac{x}{R}\exp(-n^2\pi^2\mathrm{Fo}^*)\times$$

$$\left.\times\int_0^R \varphi(x)\cos n\pi\frac{x}{R}\,dx\right]. \tag{1.7}$$

We shall now determine the temperature distribution. Let us denote the finite integral transformation of the temperature as

$$t_{c.F}(n,\tau)=\int_0^R t(x,\tau)\cos n\pi\frac{x}{R}\,dx \quad (n=0,1,2\ldots).$$

The second derivative of temperature, with consideration of conditions (XV) and (XVI), is

$$\int_0^R \frac{\partial^2 t(x,\tau)}{\partial x^2}\cos n\pi\frac{x}{R}\,dx=$$

$$=(-1)^n\frac{q_q}{\lambda_q}-\frac{n^2\pi^2}{R^2}t_{c.F}(n,\tau). \qquad (1.8)$$

Multiplying both sides of equation (XII) by $\cos n\pi\frac{x}{R}$, integrating with respect to x between the limits 0 and R, and considering equation (1.8), we get

$$\frac{dt_{c.F}(n,\tau)}{d\tau}+\frac{a_q n^2\pi^2}{R^2}t_{c.F}(n,\tau)=$$

$$=(-1)^n\frac{a_q q_q}{\lambda_q}+\frac{\varepsilon\rho c'_T}{c_q}\frac{d\theta_{c.F}(n,\tau)}{d\tau}. \qquad (1.9)$$

The value of $\frac{d\theta_{c.F}(n,\tau)}{d\tau}$ is found by differentiating (1.4) with respect to τ. After manipulations, equation (1.9) can be written as

$$\frac{dt_{c.F}(n,\tau)}{d\tau}+\frac{a_q n^2\pi^2}{R^2}t_{c.F}(n,\tau)=$$

$$=(-1)^n\frac{a_q q_q}{\lambda_q}-\frac{\varepsilon\rho c'_T}{c_q}\left[(-1)^n\frac{q_m}{\gamma_0 c'_T}\exp\left(-n^2\pi^2\frac{a_m\tau}{R^2}\right)+\right.$$

$$+\frac{a_m n^2\pi^2}{R^2}\exp\left(-n^2\pi^2\frac{a_m\tau}{R^2}\right)\times$$

$$\left.\times\int_0^R\varphi(x)\cos n\pi\frac{x}{R}\,dx\right].$$

The solution of this ordinary non-homogeneous differential equation is as follows:

$$t_{c.F} = (-1)^n \frac{q_q}{\lambda_q} \frac{R^2}{n^2\pi^2} [1 - \exp(-n^2\pi^2 \mathrm{Fo})] +$$
$$+(-1)^n \frac{\varepsilon\rho}{c_q} \frac{q_m}{\gamma_0} \frac{R^2}{n^2\pi^2(a_m - a_q)} \times$$
$$\times [\exp(-n^2\pi^2 \mathrm{Fo}^*) - \exp(-n^2\pi^2 \mathrm{Fo})] +$$
$$+ \frac{\varepsilon\rho c'_T}{c_q} \frac{a_m}{a_m - a_q} [\exp(-n^2\pi^2\mathrm{Fo}^*) - \exp(-n^2\pi^2\mathrm{Fo})] \times$$
$$\times \int_0^R \varphi(x) \cos n\pi \frac{x}{R}\, dx + C_2(n) \exp(-n^2\pi^2\mathrm{Fo}). \qquad (1.10)$$

If τ (and consequently Fo and Fo*) tend to zero, then from equation (1.10) it follows that $t_{c.F}(n, 0) = C_2(n)$. On the other hand,

$$t_{c.F}(n, 0) = \int_0^R t(x, 0) \cos n\pi \frac{x}{R}\, dx =$$
$$= \int_0^R t(x) \cos n\pi \frac{x}{R}\, dx.$$

Consequently,

$$C_2(n) = \int_0^R f(x) \cos n\pi \frac{x}{R}\, dx.$$

Using the following formula to obtain the inversion

$$t(x, \tau) = \frac{1}{R} t_{c.F}(0, \tau) + \frac{2}{R} \sum_{n=1}^{\infty} t_{c.F}(n, \tau) \cos n\pi \frac{x}{R}$$

and after some manipulations analogous to those used in obtaining $\theta(x, \tau)$, the solution for $t(x, \tau)$ can be written in the following form:

$$t(x, \tau) - \frac{1}{R} \int_0^R f(x)\, dx = \frac{q_q R}{\lambda_q} \left[\mathrm{Fo} - \frac{1}{6}\left(1 - 3\frac{x^2}{R^2}\right) + \right.$$
$$+ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n^2\pi^2} \cos n\pi \frac{x}{R} \exp(-n^2\pi^2\mathrm{Fo}) +$$

$$+\frac{2\lambda_q}{R^2 q_q}\sum_{n=1}^{\infty}\exp(-n^2\pi^2\mathrm{Fo})\cos n\pi\frac{x}{R}\times$$

$$\times\int_0^R f(x)\cos n\pi\frac{x}{R}\,dx\Big]+\frac{\varepsilon\rho}{c_q}\,\frac{\mathrm{Lu}}{\mathrm{Lu}-1}\,\frac{q_m R}{a_m\gamma_0}\times$$

$$\times\Big\{(\mathrm{Fo}-\mathrm{Fo}^*)+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n^2\pi^2}\cos n\pi\frac{x}{R}\times$$

$$\times[\exp(-n^2\pi^2\mathrm{Fo})-\exp(-n^2\pi^2\mathrm{Fo}^*)]-\frac{2\lambda_m}{R^2 q_m}\times$$

$$\times\sum_{n=1}^{\infty}[\exp(-n^2\pi^2\mathrm{Fo})-\exp(-n^2\pi^2\mathrm{Fo}^*)]\times$$

$$\times\cos n\pi\frac{x}{R}\int_0^R\varphi(x)\cos n\pi\frac{x}{R}\,dx\Big\}. \tag{1.11}$$

In obtaining the inversion (1.11) the following equality was used:

$$\frac{\varepsilon\rho}{c_q}\,\frac{q_m}{a_m\gamma_0}\,\frac{a_m\tau}{R}=$$

$$=\frac{\varepsilon\rho}{c_q}\,\frac{q_m}{a_m\gamma_0}\,\frac{a_m}{a_m-a_q}\left(\frac{a_m\tau}{R}-\frac{a_q\tau}{R}\right).$$

When the initial heat- and mass-transfer potential distributions are uniform along the plate, that is when

$$t(x,0)=t_0=\mathrm{const},$$

$$\theta(x,0)=\theta_0=\mathrm{const},$$

then from equation (1.7) and (1.11) respectively we get the following special case:

$$\Theta=\frac{\theta_0-\theta(x,\tau)}{\theta_0-\theta_p}=\mathrm{Ki}_m\Big[\mathrm{Fo}^*-\frac{1}{6}\left(1-3\frac{x^2}{R^2}\right)+$$

$$+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n^2\pi^2}\cos n\pi\frac{x}{R}\exp(-n^2\pi^2\mathrm{Fo}^*)\Big], \tag{1.12}$$

$$T=\frac{t(x,\tau)-t_0}{t_c-t_0}=\mathrm{Ki}_q\Big[\mathrm{Fo}-\frac{1}{6}\left(1-3\frac{x^2}{R^2}\right)+$$

$$+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n^2\pi^2}\cos n\pi\frac{x}{R}\exp(-n^2\pi^2\mathrm{Fo})\Big]+$$

Multiplying both parts of equation (XIII) by $rJ_0(p_n r)$, integrating with respect to r between the limits 0 and R, and considering equation (2.1), we get an ordinary differential equation with respect to τ:

$$\frac{d\,\theta_H(p_n,\tau)}{d\tau} + a_m p_n^2 \theta_H(p_n,\tau) = -\frac{q_m}{c_m \gamma_0} R J_0(p_n R).$$

The solution of this equation is

$$\theta_H(p_n,\tau) = C(p_n)\exp(-a_m p_n^2 \tau) - \\ - \frac{q_m R}{\lambda_m p_n^2} J_0(p_n R)\,[1 - \exp(-a_m p_n^2 \tau)]. \qquad (2.3)$$

The integration constant $C(p_n)$ is found using the initial condition for the mass potential. Here we get

$$C(p_n) = \int_0^R \varphi(r) r J_0(p_n r)\,dr.$$

Consequently, expression (2.3) for transformed distribution of mass-transfer potential in a cylinder can be rewritten in the following form:

$$\theta_H(p_n,\tau) = \exp(-a_m p_n^2 \tau) \int_0^R \varphi(r) r J_0(p_n r)\,dr - \\ - \frac{q_m R}{\lambda_m p_n^2} J_0(p_n R)\,[1 - \exp(-a_m p_n^2 \tau)]. \qquad (2.4)$$

The inversion $\theta_H(p_n,\tau)$ is found using the following formula:

$$\theta(r,\tau) = \frac{2}{R^2}\,\theta_H(0,\tau) + \frac{2}{R^2} \sum_{n=1}^{\infty} \theta_H(p_n,\tau) \frac{J_0(p_n r)}{J_0^2(p_n R)}.$$

Since

$$\theta_H(0,\tau) = -\frac{q_m R\tau}{c_T' \gamma_0} + \int_0^R \varphi(r) r\,dr$$

and

$$\sum_{n=1}^{\infty} \frac{2 J_0(p_n r)}{(p_n R)^2 J_0(p_n R)} = -\frac{1}{4}\left(1 - 2\frac{r^2}{R^2}\right),$$

the distribution of mass transfer potential is

$$\theta(r,\tau) - \frac{2}{R^2}\int_0^R \varphi(r) r\,dr = -\frac{q_m R}{a_m \gamma_0 c'_{\mathrm{T}}}\left[2\,\mathrm{Fo}^* - \frac{1}{4}\left(1 - 2\frac{r^2}{R^2}\right) - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{2 J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \exp\left(-\mu_n^2 \mathrm{Fo}^*\right)\right] + \frac{2}{R^2}\sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_0^2(\mu_n)} \times$$

$$\times \exp\left(-\mu_n^2 \mathrm{Fo}^*\right) \int_0^R \varphi(r) J_0\left(\mu_n \frac{r}{R}\right) dr. \tag{2.5}$$

In the above equation $\mu_n = p_n R$ are the roots of the characteristic equation

$$J_1(\mu) = 0. \tag{2.6}$$

The values of μ_n are given in Table 34.

The differential equation for temperature in a cylinder can be obtained applying to equation (XII) the finite integral Hankel transformation and considering the boundary condition (XVI):

$$\frac{dt_H(p_n,\tau)}{d\tau} + a_q p_n^2 t_H(p_n,\tau) = \frac{a_q q_q R}{\lambda_q} J_0(p_n R) +$$

$$+ \frac{\varepsilon \rho c'_{\mathrm{T}}}{c_q} \frac{d\theta_H(p_n,\tau)}{d\tau}, \tag{2.7}$$

where

$$t_H(p_n,\tau) = \int_0^R t(r,\tau)\, r\, J_0(p_n r) dr \text{ and } n = 0, 1, 2, \ldots$$

The value of $\frac{d\theta_h(p_n,\tau)}{d\tau}$ is found by differentiation of equation (2.5) with respect to τ. The value of the derivative obtained is substituted into equation (2.7). As a result, we get

$$\frac{dt_H(p_n,\tau)}{d\tau} + a_q p_n^2 t_H(p_n,\tau) = \frac{a_q q_q R}{\lambda_q} J_0(p_n R) -$$

$$- \frac{\varepsilon \rho c_{\mathrm{T}}}{c_q} \exp\left(-a_m p_n^2 \tau\right) \left\{\frac{q_m R}{\gamma_0 c'_{\mathrm{T}}} J_0(p_n R) + a_m p_n^2 \times\right.$$

$$\times \int_0^R \varphi(r)\, r\, J_0(p_n r)dr \Big\}.$$

The solution of this differential equation, after application of the corresponding initial condition, is (at $a_m \neq a_q$):

$$t_H(p_n, \tau) = \frac{q_q R}{\lambda_q p_n^2} J_0(p_n R)\,[1 - \exp(-a_q p_n^2 \tau)] +$$

$$+ \exp(-a_q p_n^2 \tau) \int_0^R f(r)\, r\, J_0(p_n r)\, d r +$$

$$+ \frac{\varepsilon \rho c_{\mathrm{T}}'}{c_q} \frac{a_m}{a_m - a_q} \Big\{ \frac{q_m R}{a_m \gamma_0 p_n^2 c_{\mathrm{T}}'} J_0(p_n R) +$$

$$+ \int_0^R \varphi(r)\, r\, J_0(p_n r)\, d r \Big\} [\exp(-a_m p_n^2 \tau) -$$

$$- \exp(-a_q p_n^2 \tau)]. \tag{2.8}$$

The inversion of equation (2.8) is found from the following:

$$t(r, \tau) = \frac{2}{R^2} t_H(0, \tau) + \frac{2}{R^2} \sum_{n=1}^{\infty} t_H(p_n, \tau) \frac{J_0(p_n r)}{J_0^2(p_n R)}$$

Since,

$$t_H(0, \tau) = \frac{a_q q_q R \tau}{\lambda_q} - \frac{\varepsilon \rho}{c_q} \frac{q_m R \tau}{\gamma_0} + \int_0^R f(r)\, r\, dr,$$

the solution of differential equation (XII), after some manipulations, is as follows:

$$t(r, \tau) - \frac{2}{R^2} \int_0^R f(r)\, r\, dr = \frac{q_q R}{\lambda_q} \Big[2\,\mathrm{Fo} - \frac{1}{4}\Big(1 - 2\frac{r^2}{R^2}\Big) -$$

$$- \sum_{n=1}^{\infty} \frac{2 J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \exp(-\mu_n^2\, \mathrm{Fo}) +$$

$$+ \frac{\lambda_q}{q_q R^3} \sum_{n=1}^{\infty} \frac{2 J_0\left(\mu_n \frac{r}{R}\right)}{J_0^2(\mu_n)} \exp(-\mu_n^2 \mathrm{Fo}) \times$$

$$\times \int_0^R f(r)\, r\, J_0\left(\mu_n \frac{r}{R}\right) dr \Bigg] + \frac{\varepsilon\rho}{c_q}\, \frac{\mathrm{Lu}}{\mathrm{Lu}-1}\, \frac{q_m R}{a_m \gamma_0} \times$$

$$\times \Bigg\{ 2\,(\mathrm{Fo}-\mathrm{Fo}^*) - \sum_{n=1}^{\infty} \frac{2\,J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0^2(\mu_n)}\, [\exp(-\mu_n^2 \mathrm{Fo}) -$$

$$-\exp(-\mu_n^2 \mathrm{Fo}^*)] - \frac{\lambda_m}{q_m R^3} \times$$

$$\times \sum_{n=1}^{\infty} \frac{2\,J_0\left(\mu_n \frac{r}{R}\right)}{J_0^2(\mu_n)}\, [\exp(-\mu_n^2 \mathrm{Fo}) - \exp(-\mu_n^2 \mathrm{Fo}^*)] \times$$

$$\times \int_0^R \varphi(r)\, r\, J_0\left(\mu_n \frac{r}{R}\right) dr \Bigg\}. \tag{2.9}$$

If the initial distribution of transfer potentials in a cylinder is uniform, i.e.,

$$f(r) = t(r, 0) = t_0 = \mathrm{const} \;\text{ and }\; \varphi(r) = \theta(r, 0) = \theta_0 = \mathrm{const},$$

then solutions of equations (2.5) and (2.9) yield, respectively,

$$\Theta = \frac{\theta_0 - \theta(r,\tau)}{\theta_0 - \theta_p} = \mathrm{Ki}_m \Bigg[2\,\mathrm{Fo}^* - \frac{1}{4}\left(1 - 2\frac{r^2}{R^2}\right) -$$

$$-\sum_{n=1}^{\infty} \frac{2\,J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \exp(-\mu_n^2 \mathrm{Fo}^*) \Bigg], \tag{2.10}$$

$$T = \frac{t(r, \quad {}^{0}}{t_c - t_0} = \mathrm{Ki}_q \Bigg[2\,\mathrm{Fo} - \frac{1}{4}\left(1 - 2\frac{r^2}{R^2}\right) -$$

$$-\sum_{n=1}^{\infty} \frac{2\,J_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \exp(-\mu_n^2 \mathrm{Fo}) \Bigg] +$$

$$+ \frac{\varepsilon\,\mathrm{Ko}\,\mathrm{Ki}_m\,\mathrm{Lu}}{\mathrm{Lu}-1} \Bigg\{ 2\,(\mathrm{Fo}-\mathrm{Fo}^*) - \sum_{n=1}^{\infty} \frac{2\,I_0\left(\mu_n \frac{r}{R}\right)}{\mu_n^2 J_0(\mu_n)} \times$$

$$\times [\exp(-\mu_n^2 \mathrm{Fo}) - \exp(-\mu_n^2 \mathrm{Fo}^*)] \Bigg\}. \tag{2.11}$$

In obtaining equations (2.10) and (2.11) the following property of the integrals was utilized:

$$\int_0^R \varphi(r)\, r\, J_0\left(\mu_n \frac{r}{R}\right) dr \text{ and } \int_0^R f(r)\, r\, J_0\left(\mu_n \frac{r}{R}\right) dr,$$

which at $p_n \neq 0$ are equal to zero.

3. SPHERE $(\delta = 0)$

$$[\Gamma = 2,\ \xi = r,\ \delta = 0,\ t(r,0) = f(r),\ \theta(r,0) = \varphi(r),$$
$$q_q = \text{const},\ q_m = \text{const}]$$

We shall now solve the system of equations for internal heat and mass transfer for a sphere with the boundary conditions considered in the preceding sections. The integration of the system will be accomplished by means of a finite integral Fourier transformation.

We shall determine the distribution of mass-transfer potential in a sphere. By applying the formula for the finite Fourier transform to equation (XIII) we obtain

$$\int_0^R \left(\frac{\partial^2 \theta(r,\tau)}{\partial r^2} + \frac{2}{R}\frac{\partial \theta(r,\tau)}{\partial r}\right) r \frac{\sin p_n r}{p_n}\, dr =$$
$$= -\frac{q_m R}{a_m \gamma_0 c'_T} \frac{\sin p_n R}{p_n} - p_n^2 \theta_{s.F}(p_n, \tau),$$

where p_n is the root of the characteristic equation

$$\operatorname{tg} pR = pR, \tag{3.1}$$

and

$$\theta_{s.F} = \int_0^R r\, \theta(r,\tau) \frac{\sin p_n r}{p_n}\, dr.$$

Therefore, the equivalent of equation (XIII) as a differential equation for the transform of θ is

$$\frac{d\theta_{s.F}(p_n,\tau)}{d\tau} + a_m\, p_n^2\, \theta_{s.F}(p_n,\tau) = -\frac{q_m R}{\gamma_0 c'_T} \frac{\sin p_n R}{p_n}. \tag{3.2}$$

The solution of the above, considering initial conditions, yields

$$\theta_{s.F}(p_n, \tau) = \exp(-a_m p_n^2 \tau) \int_0^R \varphi(r) r \frac{\sin p_n r}{p_n}\, dr -$$

$$- \frac{q_m R^2}{a_m \gamma_0 c'_{\text{T}}} \frac{\sin p_n R}{p_n R} \frac{1}{p_n^2} [1 - \exp(-a_m p_n^2 \tau)]. \qquad (3.3)$$

Applying to the above the inversion formula

$$\theta(r, \tau) = \frac{3}{R^3} \theta_{s.F}(0, \tau) +$$

$$+ \frac{2}{R} \sum_{n=1}^{\infty} \frac{p_n}{\sin^2 p_n R} \frac{\sin p_n r}{r} \theta_{s.F}(p_n, \tau),$$

we get the following expression for the mass-transfer potential:

$$\theta(r, \tau) - \frac{3}{R^3} \int_0^R \varphi(r) r^2\, dr =$$

$$= -\frac{q_m R}{c'_{\text{T}} a_m \gamma_0} \left[\frac{3 a_m \tau}{R^2} + \sum_{n=1}^{\infty} \frac{2}{p_n R \sin p_n R} \frac{\sin p_n r}{p_n r} - \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{2}{p_n R \sin p_n R} \frac{\sin p_n r}{p_n r} \exp(-a_m p_n^2 \tau) \right] +$$

$$+ \sum_{n=1}^{\infty} \frac{2}{R \sin^2 p_n R} \frac{p_n \sin p_n r}{r} \exp(-a_m p_n^2 \tau) \times$$

$$\times \int_0^R \varphi(r) r \frac{\sin p_n r}{p_n}\, dr.$$

Considering equation (3.1) and the fact that

$$\sum_{n=1}^{\infty} \frac{2}{p_n R \sin p_n R} \frac{\sin p_n r}{p_n r} = -\frac{1}{10} \left(3 - 5 \frac{r^2}{R^2} \right),$$

the solution can be written as

$$\theta(r, \tau) - \frac{3}{R^3} \int_0^R \varphi(r) r^2\, dr =$$

$$= -\frac{q_m R}{a_m \gamma_0 c_T}\left[3\,\mathrm{Fo}^* - \frac{1}{10}\left(3-5\frac{r^2}{R^2}\right) - \right.$$

$$\left. -\sum_{n=1}^{\infty}\frac{2}{\mu_n^2\cos\mu_n}\,\frac{\sin\mu_n\frac{r}{R}}{\mu_n\frac{r}{R}}\exp\left(-\mu_n^2\,\mathrm{Fo}^*\right)\right] +$$

$$+\sum_{n=1}^{\infty}\frac{2}{\mu_n\cos^2\mu_n}\,\frac{\sin\mu_n\frac{r}{R}}{Rr}\exp\left(-\mu_n^2\,\mathrm{Fo}^*\right)\times$$

$$\times\int_0^R \varphi(r)\,r\,\frac{\sin\mu_n\frac{r}{R}}{\mu_n}\,dr. \tag{3.4}$$

Here $\mu_n = p_n R$ are the roots of equation (3.1). Their values are given in Table 33.

We can now find the distribution of the heat-transfer potential in a sphere. The differential equation for the transformed temperature is

$$\frac{d\,t_{s.F}(p_n,\tau)}{d\tau} + a_q\,p_n^2 t_{s.F}(p_n,\tau) =$$

$$= \frac{a_q q_q R}{\lambda_q}\,\frac{\sin p_n R}{p_n} + \frac{\varepsilon\rho c'_T}{c_q}\,\frac{d\,\theta_{s.F}(p_n,\tau)}{d\tau}, \tag{3.5}$$

where

$$t_{s.F}(p_n,\tau) = \int_0^R t(r,\tau)\,r\,\frac{\sin p_n r}{p_n}\,dr,$$

p_n is the root of the characteristic equation (3.1).

The derivative $\frac{d\,\theta_{s.F}(p_n,\tau)}{d\tau}$ is determined from equation (3.3). Substituting the value obtained into equation (3.5), we get

$$\frac{d\,t_{s.F}(p_n,\tau)}{d\tau} + a_q\,p_n^2\,t_{s.F}(p_n,\tau) = \frac{a_q q_q R^2}{\lambda_q}\,\frac{\sin p_n R}{p_n R} -$$

$$-\exp\left(-a_m p_n^2\tau\right)\frac{\varepsilon\rho c'_T}{c_q}\left\{\frac{q_m R^2}{\gamma_0 c'_T}\,\frac{\sin p_n R}{p_n R} + \right.$$

$$\left. + a_m p_n^2\int_0^R \varphi(r)\,r\,\frac{\sin p_n r}{p_n}\,dr\right\}.$$

The solution of the above is

$$t_{s.F}(p_n, \tau) = \frac{q_q R^2}{\lambda_q} \frac{\sin p_n R}{p_n R} \frac{1}{p_n^2}[1 - \exp(-a_q p_n^2 \tau)] +$$

$$+ \exp(-a_q p_n^2 \tau) \int_0^R f(r) r \frac{\sin p_n r}{p_n} dr +$$

$$+ \frac{\varepsilon \rho c'_{\mathrm{T}}}{c_q} \frac{a_m}{a_m - a_q} \left\{ \frac{q_m R^2 \sin p_n R}{a_m \gamma_0 p_n^2 c'_{\mathrm{T}} p_n R} + \right.$$

$$\left. + [\exp(-a_m p_n^2 \tau) - \exp(-a_q p_n^2 \tau)] \int_0^R \varphi(r) r \frac{\sin p_n r}{p_n} dr \right\}.$$

The following equation can be used to obtain the inversion of the temperature transform:

$$t(r, \tau) = \frac{3}{R^3} t_{s.F}(0, \tau) +$$

$$+ \frac{2}{R} \sum_{n=1}^{\infty} \frac{p_n}{\sin^2 p_n R} \frac{\sin p_n r}{r} t_{s.F}(p_n, \tau),$$

and after some manipulations, the final solution for temperature distribution in a sphere is

$$t(r, \tau) - \frac{3}{R^3} \int_0^R f'(r) r^2 dr = \frac{q_q R}{\lambda_q} \left[3\,\mathrm{Fo} - \frac{1}{10}\left(3 - 5\frac{r^2}{R^2}\right) - \right.$$

$$- \sum_{n=1}^{\infty} \frac{2}{\mu_n^2 \cos \mu_n} \frac{\sin \mu_n \frac{r}{R}}{\mu_n \frac{r}{R}} \exp(-\mu_n^2 \mathrm{Fo}) +$$

$$+ \frac{\lambda}{q_q R} \sum_{n=1}^{\infty} \frac{2}{\mu_n \cos^2 \mu_n} \frac{\sin \mu_n \frac{r}{R}}{Rr} \exp(-\mu_n^2 \mathrm{Fo}) \times$$

$$\left. \times \int_0^R f(r) r \frac{\sin \mu_n \frac{r}{R}}{\mu_n} dr \right] + \frac{\varepsilon \rho}{c_q} \frac{\mathrm{Lu}}{\mathrm{Lu} - 1} \frac{q_m R}{a_m \gamma_0} \left\{ 3(\mathrm{Fo} - \mathrm{Fo}^*) - \right.$$

$$-\sum_{n=1}^{\infty}\frac{2}{\mu_n^2\cos\mu_n}\,\frac{\sin\mu_n\frac{r}{R}}{\mu_n\frac{r}{R}}\,[\exp(-\mu_n^2\mathrm{Fo})-\exp(-\mu_n^2\mathrm{Fo}^*)]-$$

$$-\frac{a_m\gamma_0 c_{\mathrm{T}}'}{q_m R}\sum_{n=1}^{\infty}\frac{2}{\mu_n\cos^2\mu_n}\,\frac{\sin\mu_n\frac{r}{R}}{Rr}\times$$

$$\times[\exp(-\mu_n^2\mathrm{Fo})-\exp(-\mu_n^2\mathrm{Fo}^*)]\times$$

$$\times\left.\int_0^R\varphi(r)\,r\,\frac{\sin\mu_n\frac{r}{R}}{\mu_n}\,dr\right\}. \qquad (3.6)$$

If $\varphi(r)=\theta_0=\mathrm{const}$, and $f(r)=t_0=\mathrm{const}$, then the solutions for the potentials of heat and mass transfer (3.4) and (3.6) are transformed into solutions (3.7) and (3.8) respectively:

$$\Theta=\frac{\theta_0-\theta(r,\tau)}{\theta_0-\theta_p}=\mathrm{Ki}_m\left[3\,\mathrm{Fo}^*-\frac{1}{10}\left(3-5\frac{r^2}{R^2}\right)-\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{2}{\mu_n^2\cos\mu_n}\,\frac{\sin\mu_n\frac{r}{R}}{\mu_n\frac{r}{R}}\exp(-\mu_n^2\mathrm{Fo}^*)\right], \qquad (3.7)$$

$$T=\frac{t(r,\tau)-t_0}{t_c-t_0}=\mathrm{Ki}_q\left[3\,\mathrm{Fo}-\frac{1}{10}\left(3-5\frac{r^2}{R^2}\right)-\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{2}{\mu_n^2\cos\mu_n}\,\frac{\sin\mu_n\frac{r}{R}}{\mu_n\frac{r}{R}}\exp(-\mu_n^2\mathrm{Fo})\right]+$$

$$+\frac{\varepsilon\,\mathrm{Ko}\,\mathrm{Ki}_m\,\mathrm{Lu}}{\mathrm{Lu}-1}\left\{3(\mathrm{Fo}-\mathrm{Fo}^*)-\right.$$

$$-\sum_{n=1}^{\infty}\frac{2}{\mu_n^2\cos\mu_n}\,\frac{\sin\mu_n\frac{r}{R}}{\mu_n\frac{r}{R}}\times$$

$$\left.\times[\exp(-\mu_n^2\mathrm{Fo})-\exp(-\mu_n^2\mathrm{Fo}^*)]\right\}. \qquad (3.8)$$

4. ANALYSIS OF SOLUTIONS.

We can now analyse the solutions obtained in Sections 1 to 3 (equations (1.7), (1.11)-(1.13), (2.5), (2.9) to (2.11), (3.4) and (3.6) to (3.8)). In the case of uniform initial distributions of transfer potentials, the distribution of temperature represents the superposition of three temperature fields. The first field, described by the expression in brackets, characterizes the heating of the material without considering either mass transfer or phase transformation in the body. The second field, i.e. terms containing Fo, takes into consideration the effect of stationary heat sinks (sources). Finally, the third field, i.e. terms containing Fo*, describes the effect of a moving heat source, resulting from phase transformations in moving mass.

In the case when $\varepsilon = 0$, i.e. no phase transformations, all three solutions reduce to the usual classical solutions of heat transfer without a heat source.

The above considerations also apply to the more general solutions that are obtained on the assumption that the initial temperature and mass potential distributions are not uniform but are given as functions of the coordinates. In the latter case the solutions contain additional terms which describe the effect of the initial conditions.

The series appearing in the solutions converge rapidly and therefore the solutions are convenient for engineering calculations.

The distribution of the mass transfer potential corresponds to classical solutions of diffusion and conduction equations. As time increases, or more exactly as Fo* increases, the terms of the series decrease rapidly and at a certain value of $\mathrm{Fo}^* > \mathrm{Fo}_1^*$, become exceedingly small in comparison with the first two terms. Thus the entire series can be neglected. Beginning from this time, $\theta(\xi, \tau)$ at any point of the body becomes a linear function of time and its distribution in the body becomes parabolic, i.e., a quasi-stationary regime is established.

5. INFINITE CYLINDER ($\delta \neq 0$)

$$[\Gamma = 1, \quad \xi = r, \quad t(r, 0) = f_1(r), \quad \theta(r, 0) = f_2(r),$$

$$q_q = q_q(\tau), \quad q_m = q_m(\tau)]$$

We shall solve the system of equations (I') to (II') with boundary conditions (IX) to (XI) and initial conditions (XIV) by simultaneous application of integral Hankel and Laplace transformations. In order to simplify the notation used in this section we shall denote the heat and mass transfer potentials by $\theta_i (i = 1,2)$ so that $\theta_1 = t$ and $\theta_2 = \theta$. In addition, the subscripts q and m will be denoted by 1 and 2 respectively.

We use the integral Hankel transformation with respect to the variable r, i.e.,

$$\{\theta_i(p,\tau)\}_H = \int_0^R r\,\theta_i(r,\tau)\,J_0(pr)\,dr, \tag{5.1}$$

where p is the root of the characteristic equation $J_0'(pR)=0$.

The inversion, $\theta_i\,(r,\tau)$, of the transform $\{\theta_i\,(p,\tau)\}_H$ is obtained from the following formula:

$$\theta_i\,(r,\tau) = \frac{2}{R^2}\int_0^R r\,\theta_i\,(r,\tau)\,dr +$$

$$+\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(p_n r)}{J_0^2\,(p_n R)}\{\theta_i\,(p,\tau)\}_H\,. \tag{5.2}$$

The application of the Hankel transformation (5.1) with consideration of boundary conditions (IX)-(XI) to the group of terms contained in brackets in equations (I')-(II') leads to the following relationships:

$$\int_0^R\left[\frac{\partial^2\theta_1(r,\tau)}{\partial r^2}+\frac{1}{r}\,\frac{\partial\theta_1(r,\tau)}{\partial r}\right]r\,J_0(pr)\,dr =$$

$$= R\,\frac{q_1(\tau)}{\lambda_1}\,J_0(pR) - p^2\{\theta_1\,(p,\tau)\}_H\,, \tag{5.3}$$

$$\int_0^R\left[\frac{\partial^2\theta_2(r,\tau)}{\partial r^2}+\frac{1}{r}\,\frac{\partial^2\theta_2(r,\tau)}{\partial r}\right]rJ_0\,(pr)\,dr =$$

$$= -R\left[\delta\,\frac{q_1(\tau)}{\lambda_1}+\frac{q_2(\tau)}{\lambda_2}\right]J_0(pR) - p^2\{\theta_2\,(p,\tau)\}_H\,. \tag{5.4}$$

We assume that the operator of the Hankel transformation $\{\ \}_H$ is commutative with the operator of differentiation $\frac{\partial}{\partial\tau}$. Then, following multiplication of all terms of equations (I')-(II') by $rJ_0\,(pr)$ and integrating with respect to r from 0 to R and considering equations (5.3) to (5.4), the original system of partial differential equations is transformed to a system of two simple differential equations.

$$\left.\begin{aligned}
&\frac{d}{d\tau}\{\theta_1(p,\tau)\}_H = a_1\Big[R\frac{q_1(\tau)}{\lambda_1}J_0(pR)-\\
&-p^2\{\theta_1(p,\tau)\}_H\Big]+\varepsilon\rho\frac{c'}{c}\frac{d}{d\tau}\{\theta_2(p,\tau)\}_H\\
&\frac{d}{d\tau}\{\theta_2(p,\tau)\}_H = -a_2\Big[R\frac{q_2(\tau)}{\lambda_2}J_0(pR)+\\
&+p^2\{\theta_2(p,\tau)\}_H\Big]-a_2\delta p^2\{\theta_1(p,\tau)\}_H
\end{aligned}\right\}, \tag{5.5}$$

In the derivation of equations (5.5) conditions (XIV) are used in the following form, obtained by application of the Hankel transformation:

$$[\theta_i(p,0)\}_H = \{f_i(p)\}_H \quad (i=1,2). \tag{5.6}$$

We can now apply to the system of differential equations (5.5) and initial conditions (5.6):

$$\{\theta_i(r,s)\}_L = s\int_0^\infty \theta_i(r,\tau)\,e^{-s\tau}d\tau, \tag{5.7}$$

and as a result we obtain the following system of algebraic equations:

$$(s+a_1p^2)\{\theta_l(p,s)\}_{HL}-\frac{\varepsilon\rho c'}{c}s\{\theta_2(p,s)\}_{HL}=$$

$$=s\{f_1(p)\}_H+\frac{a_1}{\lambda_1}RJ_0(pR)\{q_1(s)\}_L-\frac{\varepsilon\rho c'}{c}s\{f_2(p)\}_H,$$

$$a_2\delta p^2\{\theta_1(p,s)\}_{HL}+(s+a_2p^2)\{\theta_2(p,s)\}_{HL}=$$

$$=s\{f_2(p)\}_H-\frac{a_2}{\lambda_2}RJ_0(pR)\{q_2(s)\}_L.$$

Solving the above system for $\{\theta_i(p,s)\}_{HL}$ $(i=1,2)$ and obtaining the inversion of these potentials with respect to s, we get

$$\{\theta_i(p,\tau)\}_H=\sum_{j,k=1}^{2}[A^i_{jk}\{P_{jk}\}_H+B^i_{jk}\{Q_{jk}\}_H], \tag{5.8}$$

where

$$A^1_{j1} = (-1)^j \frac{1-\nu_j^2}{\nu_1^2-\nu_2^2}, \quad A^1_{j2} = (-1)^j \frac{\varepsilon\rho\frac{c'}{c}}{\nu_2^2-\nu_1^2},$$

$$B^1_{j1} = (-1)^j \frac{a_1}{\lambda_1} \frac{1-\nu_j^2}{\nu_1^2-\nu_2^2},$$

$$B^1_{j2} = (-1)^j \nu_j^2 \frac{a_2}{\lambda_2} \frac{\varepsilon\rho\frac{c'}{c}}{\nu_1^2-\nu_2^2} = (-1)^j \frac{a_2}{\lambda_2\delta} - \frac{\varepsilon\,\mathrm{KoPn}\,\nu_j^2}{\nu_1^2-\nu_2^2},$$

$$A^2_{j1} = (-1)^j \frac{\delta}{\nu_2^2-\nu_1^2}, \quad A^2_{j\,2} = (-1)^j \left[\frac{\frac{1}{\mathrm{Lu}} - \nu_j^2 + \varepsilon\,\mathrm{KoPn}}{\nu_1^2-\nu_2^2}\right],$$

$$B^2_{j1} = (-1)^j \frac{a_1}{\lambda_1} \frac{\delta}{\nu_2^2-\nu_1^2}, \quad B^2_{j2} = (-1)^j \frac{a_2}{\lambda_2} \frac{\frac{1}{\mathrm{Lu}} - \nu_j^2}{\nu_1^2-\nu_2^2},$$

$$\nu_j^2 = \frac{1}{2}\left[\left(1+\varepsilon\,\mathrm{KoPn}+\frac{1}{\mathrm{Lu}}\right) + \right.$$

$$\left. + (-1)^{j+1}\sqrt{\left(1+\varepsilon\,\mathrm{KoPn}+\frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}}\,\right],$$

$$\{P_{jk}\}_H = \{f_k(p)\}_H \exp\left(-R^2p^2\nu_j^2\,\mathrm{Fo}_2\right),$$

$$\{Q_{jk}\}_H = RJ_0(pR)\int_0^\tau q_k(\eta)\times$$

$$\times\exp\left[-R^2p^2\nu_j^2\,\frac{a_2(\tau-\eta)}{R^2}\right]d\eta,$$

$$\mathrm{Fo}_2 = \frac{a_2\tau}{R^2} = \mathrm{LuFo}, \quad (j=1,2).$$

It is easily checked that the function $\{\theta_i(p,\tau)\}_H$ determined by expression (5.8) satisfies the initial conditions (5.6).

The inversion of $\{\theta_i(p,\tau)\}_H$ can now be determined. For this purpose we use equation (5.2) which after some manipulations yields

$$\theta_i(r, \tau) = \sum_{j,\,k=1}^{2} (A^i_{jk} P_{jk} + B^i_{jk} Q_{jk}) \quad (i=1,2),$$

where

$$P_{jk} = \frac{2}{R^2}\left[\int_0^R r f_k(r)\,dr + \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_0^2(\mu_n)} \times\right.$$

$$\left.\times \exp(-\mu_n^2 \nu_j^2 \mathrm{Fo}_2) \int_0^R r f_k(r)\, J_0\left(\mu_n \frac{r}{R}\right) dr\right],$$

$$Q_{jk} = \frac{2}{R}\left[\int_0^{\tau} q_k(\eta)\,d\eta + \sum_{n=1}^{\infty} \frac{J_0\left(\mu_n \frac{r}{R}\right)}{J_0(\mu_n)} \times\right.$$

$$\left.\times \exp(-\mu_n^2 \nu_j^2 \mathrm{Fo}_2) \int_0^R q_k(\eta) \exp\left(\mu_n^2 \nu_j^2 \frac{a_2\eta}{R^2}\right)\right] d\eta,$$

μ_n are the roots of the equation $J_0'(\mu)=0$.

6. SPHERE ($\delta \neq 0$)

$$[\Gamma=2,\ \xi=r,\ t(r,\ 0)=f_1(r),\ \theta(r,\ 0)=f_2(r),$$

$$q_q=q_q(\tau),\ q_m^{*}=q_m(\tau)]$$

The problem is solved using the same conditions as employed in the preceding section. Here we utilize simultaneous integral Fourier sine and Laplace transformations. The Fourier sine transformation noted in Section 3 has the following form:

$$\{\theta_i(p,\ \tau)\}_F = \int_0^R r\,\theta_i(r,\ \tau)\,\frac{\sin pr}{p}\,dr, \tag{6.1}$$

where p is the root of the characteristic equation

$$\mathrm{tg}\, pR = pR.$$

Analogous to the procedure employed in Sections 3 and 5, we apply to the system of equations (I') to (II') and boundary conditions (IX) to (XI) and (XIV), first the transformation (6.1) and then the Laplace transformation (5.7). We get the following system of algebraic equations:

$$(s+a_1p^2)\{\theta_1(p,\ s)\}_{FL}-\frac{\varepsilon\rho c'}{c}s\{\theta_2(p,\ s)\}_{FL}=$$

$$=s\{f_1(p)\}_F+R\frac{a_1}{\lambda_1}\frac{\sin pR}{p}\{q_1(s)\}_L-$$

$$-\frac{\varepsilon\rho c'}{c}s\{f_2(p)\}_F, \tag{6.2}$$

$$a_2\delta p^2\{\theta_1(p,\ s)\}_{FL}+(s+a_2p^2)\{\theta_2(p,\ s)\}_{FL}=$$

$$=s\{f_2(p)\}_F-R\frac{a_2}{\lambda_2}\frac{\sin pR}{p}\{q_2(s)\}_L. \tag{6.3}$$

Solving the above for $\{\theta_i(p,\ s)\}_{FL}$ $(i=1,2)$ and obtaining the inversions with respect to s, we get

$$\{\theta_i(p,\ \tau)\}_F=\sum_{j,\,k=1}^{2}[A^i_{jk}\{P_{jk}\}_F+B^i_{jk}\{Q_{jk}\}_F], \tag{6.4}$$

where A^i_{jk} and B^i_{jk} are determined by expressions given in the preceding section:

$$\{P_{jk}\}_F=\{f_k(p)\}_F\exp[-R^2p^2\nu_j^2\,\mathrm{Fo}_2],$$

$$\{Q_{jk}\}_F=\frac{R\sin pR}{p}\int_0^{\tau}q_k(\eta)\exp\left[p^2R^2\nu_j^2\frac{a_2(\tau-\eta)}{R^2}\right]d\eta.$$

The inverse Fourier transformation for the case at hand can be obtained from the following formula:

$$\theta_i(r,\ \tau)=\frac{3}{R^3}\int_0^R r^2f_i(r,\ \tau)\,dr+$$

$$+\frac{2}{R}\sum_{n=1}^{\infty}\frac{p_n}{\sin^2 p_nR}\frac{\sin p_nr}{r}\{\theta_i(p,\ \tau)\}_F. \tag{6.5}$$

Applying equation (6.5) to equation (6.4), we finally get the following heat and mass transfer potentials:

$$\theta_l\ (r,\ \tau) = \sum_{j,\ k=1}^{2} (A^l_{jk} P_{jk} + B^l_{jk} Q_{jk}),$$

where

$$P_{jk} = \frac{3}{R^3} \int_0^R r^2 f_k\ (r)\, dr + \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\mu_n}{\sin^2 \mu_n} \times$$

$$\times \frac{\sin \mu_n \frac{r}{R}}{\frac{r}{R}} \exp\left(-\mu_n^2\ \nu_j^2\, \mathrm{Fo}_2\right) \int_0^R r f_k\,(r)\, \frac{\sin \mu_n \frac{r}{R}}{\mu_n}\, dr,$$

$$Q_{jk} = \frac{3}{R^2} \int_0^{\tau} q_k\ (\eta)\, d\,\eta + \frac{2}{R} \sum_{n=1}^{\infty} \frac{\sin \mu_n \frac{r}{R}}{\frac{r}{R} \sin \mu_n} \times$$

$$\times \exp\left(-\mu_n^2\ \nu_j^2\, \mathrm{Fo}_2\right) \int_0^{\tau} q_k\,(\eta) \exp\left(\mu_n^2\ \nu_j^2\, \frac{a_2 \eta}{R^2}\right) d\,\eta,$$

and μ_n are the roots of the equation

$$\mathrm{tg}\, \mu = \mu.$$

7. INFINITE PLATE. GENERALIZED BOUNDARY CONDITIONS OF THE SECOND KIND.

We solve the system of equations (I')-(II') with initial conditions (XIV), conditions of symmetry (XV) and with boundary conditions of the following form:

$$\frac{\partial\, \theta_1 (R,\, \tau)}{\partial x} + \alpha\, \theta_1 (R,\, \tau) + \beta\, \theta_2 (R,\, \tau) = \varphi_1 (\tau), \tag{7.1}$$

$$\frac{\partial\, \theta_2 (R,\, \tau)}{\partial x} + \alpha' \frac{\partial\, \theta_1 (R,\, \tau)}{\partial x} + \beta'\, \theta_2 (R,\, \tau) = \varphi_2 (\tau). \tag{7.2}$$

Here as in the preceding sections, $\theta_1 = t$, $\theta_2 = \theta$, α, β, α', and β' are known.

In accordance with the work of A. P. Prudnikov we consider the auxiliary problem in which the boundary conditions (7.1) and (7.2) are replaced by

$$\theta_k(R, \tau) = \chi_k(\tau), \qquad (k = 1,2). \tag{7.3}$$

The latter are solved by means of simultaneous application of integral Fourier and Laplace transformations. Applying the finite Fourier cosine transformation with respect to the variable x, we get

$$\{\theta_i(p, \tau)\}_F = \int\limits_0^R \theta_i(x, \tau) \cos\left(n + \frac{1}{2}\right)\frac{\pi}{R}\, x dx. \tag{7.4}$$

The inverse transformation of transfer potentials $\{\theta_i(p, \tau)\}_F$ is determined from the following:

$$\theta_i(x, \tau) = \frac{2}{R}\sum_{n=1}^{\infty} \{\theta_i(p, \tau)\}_F \cos\left(n + \frac{1}{2}\right)\frac{\pi}{R}\, x. \tag{7.5}$$

The Laplace transformations are applied with respect to the variable τ :

$$\{\theta_i(x, s)\}_L = \int\limits_0^{\infty} e^{-s\tau}\theta(x, \tau)\, d\tau. \tag{7.6}$$

Successive application of integral transformations (7.4) and (7.6), together with initial conditions (XIV) and boundary conditions (7.3) reduces the original problem to an algebraic system. Then, by application of the inverse Laplace transformation, we obtain

$$\{\theta_i(p, \tau)\}_F = \sum_{j,\, s=1}^{2} [A^i_{js}\{v_{js}\}_F + B^i_{js}\{w_{js}\}_F] \; (i = 1,2), \tag{7.7}$$

where

$$\{v_{js}\}_F = \exp\left[-\left(n + \frac{1}{2}\right)^2 \pi^2 \nu_j^2 \mathrm{Fo}_2\right] \{f_s(p)\}_F,$$

$$\{w_{js}\}_F = (-1)^n \left(n + \frac{1}{2}\right)\frac{\pi}{R} \times$$

$$\times \exp\left[-\left(n + \frac{1}{2}\right)^2 \pi^2 \nu_j^2 \mathrm{Fo}_2\right]^{\tau}_{*} \chi_s(\tau),$$

$$\nu_j^2 = \frac{1}{2}\left[\left(1 + \varepsilon\mathrm{KoPn} + \frac{1}{\mathrm{Lu}}\right) + (-1)^{j+1} \times \right.$$
$$\left. \times \sqrt{\left(1 + \varepsilon\mathrm{KoPn} + \frac{1}{\mathrm{Lu}}\right)^2 - \frac{4}{\mathrm{Lu}}}\;\right],$$

$$\mathrm{Fo}_2 = \frac{a_2 \tau}{R^2}.$$

The sign $\overset{\tau}{*}$ denotes the convolution operation

$$f(\tau) \overset{\tau}{*} g(\tau) = \int_0^\tau f(\eta) g(\tau - \eta) d\eta = \int_0^\tau f(\tau - \eta) g(\eta) d\eta = g(\tau) \overset{\tau}{*} f(\tau).$$

The constants A^i_{js}, B^i_{js} (i, j, s = 1.2) are determined from the following:

$$A^1_{j1} = (-1)^j \frac{1 - \nu_j^2}{\nu_1^2 - \nu_2^2}, \qquad A^1_{j2} = (-1)^j \frac{\varepsilon \rho \dfrac{c'}{c}}{\nu_2^2 - \nu_1^2},$$

$$B^1_{j1} = (-1)^j \frac{a_1 \left[1 - \nu_j^2 - \varepsilon \mathrm{KoPnLu}\, \nu_j^2\right]}{\nu_1^2 - \nu_2^2},$$

$$B^1_{j2} = (-1)^j \frac{a_2}{\delta}\, \varepsilon\, \mathrm{KoPn} \frac{\nu_j^2}{\nu_2^2 - \nu_1^2},$$

$$A^2_{j1} = (-1)^j \frac{\delta}{\nu_2^2 - \nu_1^2},$$

$$A^2_{j2} = (-1)^j \frac{\nu_j^2 - \varepsilon\, \mathrm{KoPn} - \dfrac{1}{\mathrm{Lu}}}{\nu_2^2 - \nu_1^2},$$

$$B^2_{j1} = (-1)^j a_2 \delta \frac{\nu_j^2}{\nu_2^2 - \nu_1^2},$$

$$B^2_{j2} = (-1)^j \frac{a_2\left(\nu_j^2 - \dfrac{1}{\mathrm{Lu}}\right)}{\nu_2^2 - \nu_1^2}.$$

Application of equation (7.5) to equation (7.7) yields

$$\theta_i (x, \tau) = \sum_{j,s=1}^{2} \left(A^i_{js} v_{js} + B^i_{js} w_{js}\right) \quad (i = 1,\ 2), \tag{7.8}$$

where

$$v_{js} = \frac{2}{R}\sum_{n=0}^{\infty} \exp\left[-\left(n+\frac{1}{2}\right)^2 \pi^2 \nu_j^2 \mathrm{Fo}_2\right] \times$$

$$\times \int_0^R f_s(\xi) \cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}\,\xi\, d\xi \cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}x, \tag{7.9}$$

$$w_{js} = \frac{2}{R}\sum_{n=0}^{\infty} \exp\left[-\left(n+\frac{1}{2}\right)^2 \pi^2 \nu_j^2 \mathrm{Fo}_2\right] \overset{\tau}{*} (-1)^n \times$$

$$\times \left(n+\frac{1}{2}\right)\frac{\pi}{R}\,\chi_s(\tau) \cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}x. \tag{7.10}$$

Equations (7.9) and (7.10) can be simplified by introducing the theta-function. After some manipulations, we get

$$v_{js} = \frac{1}{2R}\int_0^R \left\{\vartheta_2\left[\frac{x-\xi}{2R},\ \nu_j^2 \mathrm{Fo}_2\right] + \vartheta_2\left[\frac{x+\xi}{2R},\ \nu_j^2 \mathrm{Fo}_2\right]\right\} \times$$

$$\times f_s(\xi)\, d\xi,$$

$$w_{js} = \frac{1}{R}\int_0^{\tau} \frac{\partial}{\partial x}\vartheta_2\left[\frac{R-x}{2R},\ \nu_j^2\frac{a_2(\tau-\eta)}{R^2}\right]\chi_s(\eta)\, d\eta, \tag{7.11}$$

$$\vartheta_2(x,\tau) = \frac{1}{\sqrt{\pi\tau}}\sum_{n=-\infty}^{\infty}(-1)^n \exp\left[-\frac{(x+n)^2}{\tau}\right]. \tag{7.12}$$

In order to solve the problem considered in this section, it is necessary to determine certain unknown functions $\chi_k(\tau)$ from boundary conditions (7.1) and (7.2). Differentiating (7.8) with respect to x and replacing the second derivative of ϑ_2 with respect to x by its derivative with respect to τ, we get

$$\frac{\partial \theta_i(x,\tau)}{\partial x} = \sum_{j,s=1}^{2}(A_{js}^i P_{js} + B_{js}^i Q_{js}),\ (i=1,2), \tag{7.13}$$

where

$$P_{js}(x,\tau) = \frac{1}{2R}\int_0^R \frac{\partial}{\partial x}\left\{\vartheta_2\left[\frac{x-\xi}{2R},\ \nu_j^2 \mathrm{Fo}_2\right] + \right.$$

$$+ \vartheta_2 \left[\frac{x+\xi}{2R}, \ \nu_j^2 \,\mathrm{Fo}_2 \right] \Big\} f_s(\xi)\, d\xi,$$

$$Q_{js}(x\,\tau) = \frac{1}{\nu_j^2 a_2 R} \int_0^\tau \frac{\partial}{\partial(\tau-\eta)} \vartheta_2 \left[\frac{R-x}{2R}, \ \nu_j^2 \times \right.$$

$$\left. \times \frac{a_2(\tau-\eta)}{R^2} \right] \chi_s(\eta)\, d\eta. \qquad (7.14)$$

It is easy to show that the derivative $\frac{\partial}{\partial(\tau-\eta)}\vartheta_2$, which exists for all values of $x \neq R$, tends to infinity as $\tau^{-3/2}$ as $\tau \to 0$, when $x = R$ and $\tau \to \eta$. We therefore replace boundary conditions (7.1) and (7.2) by conditions obtained by integration with respect to τ. In this case we get

$$\int_0^\tau \frac{\partial \theta_1(R,\eta)}{\partial x} d\eta + \alpha \int_0^\tau \chi_1(\eta)\, d\eta + \beta \int_0^\tau \chi_2(\eta)\, d\eta =$$

$$= \int_0^\tau \varphi_1(\eta)\, d\eta, \qquad (7.15)$$

$$\int_0^\tau \frac{\partial \theta_2(R,\eta)}{\partial x} d\eta + \alpha' \int_0^\tau \frac{\partial \theta_1(R,\eta)}{dx} d\eta + \beta' \int_0^\tau \chi_2(\eta)\, d\eta =$$

$$= \int_0^\tau \varphi_2(\eta)\, d\eta. \qquad (7.16)$$

Integration of (7.13) with respect to τ yields

$$\int_0 \frac{\partial \theta_i(x,\eta)}{dx} d\eta = \sum_{j,s=1} (A_{js}^i P_{js}^* + B_{js}^i Q_{js}^*) \ (i = 1,2), \qquad (7.17)$$

where

$$P_{js}^*(x,\tau) = \int_0^\tau P_{js}(x,\eta)\, d\eta, \quad Q_{js}^*(x,\tau) = \int_0^\tau Q_{js}(x,\eta)\, d\eta.$$

Integrating equation (7.14) with respect to τ and changing the order of integration by means of the Dirichlet formula, we get

$$Q^*_{js}(x, T) = \frac{1}{\nu_j^2 a_2 R} \int_0^T d\tau \int_0^\tau \chi_s(\eta) \frac{\partial}{\partial(\tau-\eta)} \times$$

$$\times \vartheta_2 \left[\frac{R-x}{2R}, \nu_j^2 \frac{a_2(\tau-\eta)}{R^2} \right] d\eta =$$

$$= \frac{1}{\nu_j^2 a_2 R} \int_0^T \chi_s(\eta)\, d\eta \int_\eta^T \frac{\partial}{\partial(\tau-\eta)} \vartheta_2 \left[\frac{R-x}{2R}, \nu_j^2 \times \right.$$

$$\left. \times \frac{a_2(\tau-\eta)}{R^2} \right] d\tau = \frac{1}{\nu_j^2 a_2 R} \times$$

$$\times \int_0^T \chi_s(\eta)\, d\eta \left\{ \vartheta_2 \left[\frac{R-x}{2R}, \nu_j^2 \frac{a_2(\tau-\eta)}{R^2} \right] - \vartheta_2 \left[\frac{R-x}{2R}, 0 \right] \right\}.$$

Whence it follows that for all values of $x < R$,

$$Q^*_{js}(x, T) = \frac{1}{\nu_j^2 a_2 R} \int_0^T \chi_s(\eta)\, \vartheta_2 \left[\frac{R-x}{2R}, \nu_j^2 \times \right.$$

$$\left. \times \frac{a_2(T-\eta)}{R^2} \right] d\eta. \qquad (7.18)$$

Analysis shows that equation (7.18) also applies at $x = R$, i.e.,

$$Q^*_{js}(R, T) = \frac{1}{\nu_j^2 a_2 R} \int_0^T \chi_s(\eta) \vartheta_2 \left[0, \nu_j^2 \frac{a_2(T-\eta)}{R^2} \right] d\eta. \qquad (7.19)$$

The latter integral converges because according to equation (7.12), the function ϑ_2 tends to infinity as $\eta \to T$ and $T \to 0$. Using equations (7.17) to (7.19) to obtain the corresponding terms of the modified boundary conditions (7.15) and (7.16), and also taking into account equation (7.3), we get a system of Volterra integral equations of the first kind:

$$\sum_{\beta=1}^{2} \int_0^\tau K_{\alpha\beta}(\tau-\eta)\, \chi_\beta(\eta)\, d\eta = g_\alpha(\tau) \quad (\alpha = 1, 2), \qquad (7.20)$$

where

$$K_{11}(\tau-\eta)=\frac{B_{11}^{1}}{\nu_1^2 a_2 R}\vartheta_2\left[0,\ \nu_1^2\frac{a_2(\tau-\eta)}{R^2}\right]+$$

$$+\frac{B_{21}^{1}}{\nu_2^2 a_2 R}\vartheta_2\left[0,\ \nu_2^2\frac{a_2(\tau-\eta)}{R^2}\right]+\alpha,$$

$$K_{12}(\tau-\eta)=\frac{B_{12}^{1}}{\nu_1^2 a_2 R}\vartheta_2\left[0,\ \nu_1^2\frac{a_2(\tau-\eta)}{R^2}\right]+$$

$$+\frac{B_{22}^{1}}{\nu_2^2 a_2 R}\vartheta_2\left[0,\ \nu_2^2\frac{a_2(\tau-\eta)}{R^2}\right]+\beta,$$

$$K_{21}(\tau-\eta)=\frac{\alpha' B_{11}^{1}+B_{11}^{2}}{\nu_1^2 a_2 R}\vartheta_2\left[0,\ \nu_1^2\frac{a_2(\tau-\eta)}{R^2}\right]+$$

$$+\frac{\alpha' B_{21}^{1}+B_{21}^{2}}{\nu_2^2 a_2 R}\vartheta_2\left[0,\ \nu_2^2\frac{a_2(\tau-\eta)}{R^2}\right],$$

$$K_{22}(\tau-\eta)=\frac{\alpha' B_{12}^{1}+B_{12}^{2}}{\nu_1^2 a_2 R}\vartheta_2\left[0,\ \nu_1^2\frac{a_2(\tau-\eta)}{R^2}\right]+$$

$$+\frac{\alpha' B_{22}^{1}+B_{22}^{2}}{\nu_2^2 a_2 R}\vartheta_2\left[0,\ \nu_2^2\frac{a_2(\tau-\eta)}{R^2}\right]+\beta',$$

$$g_1(\tau)=\int_0^{\tau}\varphi_1(\eta)\,d\eta-\sum_{j,s=1}^{2}A_{js}^{1}P_{js}^{*}(R,\tau),$$

$$g_2(\tau)=\int_0^{\tau}\varphi_2(\eta)\,d\eta-\sum_{j,s=1}^{2}(\alpha' A_{js}^{1}+A_{js}^{2})\,P_{js}^{*}(R,\tau).$$

Here we have defined

$$K_{11}(\tau-\eta)=\frac{G_1(\tau-\eta)}{\sqrt{\tau-\eta}},\quad K_{22}(\tau-\eta)=\frac{G_2(\tau-\eta)}{\sqrt{\tau-\eta}},\qquad (7.21)$$

where

$$G_1(\tau-\eta)=\frac{\varkappa_1}{\sqrt{\pi}}+\frac{B_{11}^{1}}{\nu_1^2 a_2\sqrt{\nu_1^2 a_2\pi}}\sum_{n=-\infty}^{\infty}{}'(-1)^n\times$$

$$\times \exp\left[-\frac{R^2n^2}{\nu_1^2 a_2(\tau-\eta)}\right] + \frac{B_{21}^1}{\nu_2^2 a_2\sqrt{\nu_2^2 a_2 \pi}} \sum_{n=-\infty}^{\infty}{}' (-1)^n \times$$

$$\times \exp\left[-\frac{R^2n^2}{\nu_2^2 a_2(\tau-\eta)}\right] + \alpha\sqrt{\tau-\eta},$$

$$G_2(\tau-\eta) = \frac{\varkappa_2}{\sqrt{\pi}} + \frac{\alpha' B_{22}^1 + B_{12}^2}{\nu_1^2 a_2\sqrt{\nu_1^2 a_2 \pi}} \sum_{n=-\infty}^{\infty}{}' (-1)^n \times$$

$$\times \exp\left[-\frac{R^2n^2}{\nu_1^2 a_2(\tau-\eta)}\right] + \frac{\alpha' B_{22}^1 + B_{22}^2}{\nu_2^2 a_2\sqrt{\nu_2^2 a_2 \pi}} \sum_{n=-\infty}^{\infty}{}' (-1)^n \times$$

$$\times \exp\left[-\frac{R^2n^2}{\nu_2^2 a_2(\tau-\eta)}\right] + \beta'\sqrt{\tau-\eta},$$

$$\varkappa_1 = \frac{B_{11}^1}{\nu_1^2 a_2\sqrt{\nu_1^2 a_2}} + \frac{B_{21}^1}{\nu_2^2 a_2\sqrt{\nu_2^2 a_2}},$$

$$\varkappa_2 = \frac{\alpha' B_{12}^1 + B_{12}^2}{\nu_1^2 a_2\sqrt{\nu_1^2 a_2}} + \frac{\alpha' B_{22}^1 + B_{22}^2}{\nu_2^2 a_2\sqrt{\nu_2^2 a_2}}$$

The summation Σ' in these expressions is over all integer values of n, with the exception of $n=0$. Substituting expression (7.21) into (7.20) we get a system of generalized Abel integral equations which can be reduced to the system of Volterra integral equations of the second kind.

$$\varphi_\alpha(\tau) + \sum_{\beta=1}^{2}\int_0^\tau \varphi_{\alpha\beta}(\tau,\eta)\varphi_\beta(\eta)\,d\eta = \omega_\alpha(\tau) \quad (\alpha = 1, 2), \tag{7.22}$$

where

$$\varphi_{\alpha\beta}(\tau,\eta) = \frac{1}{\varkappa_\alpha\sqrt{\pi}}\,\frac{d}{d\tau}L_{\alpha\beta}(\tau,\eta),$$

$$\omega_\alpha(\tau) = \frac{1}{\varkappa_\alpha\sqrt{\pi}}\,\frac{d}{d\tau}s_\alpha(\tau),$$

$$L_{\alpha\beta}(\tau,\eta) = \int_\eta^\tau \frac{G_\alpha(T-\eta)}{\sqrt{(\tau-T)(T-\eta)}}\,dT \quad (\alpha=\beta),$$

$$L_{\alpha\beta}(\tau, \eta) = \int_{\eta}^{\tau} \frac{K_{\alpha\beta}(T-\eta)}{\sqrt{\tau - T}}\, dT \quad (\alpha \neq \beta),$$

$$s_{\alpha}(\tau) = \int_{0}^{\tau} \frac{g_{\alpha}(T)}{\sqrt{\tau - T}}\, dT,$$

α and β have values of one and two, respectively. The final solution of this problem is obtained following the determination of $\varphi_{\alpha}(\tau)$ from the system of equations (7.22) and substitution into equations (7.8) and (7.11).

CHAPTER V

Two-Dimensional Fields of Heat and Mass Transfer Potentials

1. FINITE PLATE.
BOUNDARY CONDITIONS OF THE SECOND KIND.

Let us consider the solution of the differential heat and mass transfer equations as applied to a two-dimensional plate. We have

$$\frac{\partial t(x,y,\tau)}{\partial\tau}=a_q\left(\frac{\partial^2 t(x,y,\tau)}{\partial x^2}+\frac{\partial^2 t(x,y,\tau)}{\partial y^2}\right)+$$

$$+\frac{\varepsilon\rho}{c_q}\,\frac{\partial u(x,y,\tau)}{\partial\tau},\tag{1.1}$$

$$\frac{\partial u(x,y,\tau)}{\partial\tau}=a_m\left(\frac{\partial^2 u(x,y,\tau)}{\partial x^2}+\frac{\partial^2 u(x,y,\tau}{\partial y^2}\right)+$$

$$+a_m\delta\left(\frac{\partial^2 t(x,y,\tau)}{\partial x^2}+\frac{\partial^2 t(x,y,\tau)}{\partial y^2}\right),\tag{1.2}$$

where $u=c'_T\,\theta$ is the mass concentration.

The initial conditions are

$$t(x,y,0)=f(x,y),\quad u(x,y,0)=\varphi(x,y).\tag{1.3}$$

The boundary conditions are

$$\frac{\partial t(0,y,\tau)}{\partial x}=\frac{\partial t(x,0,\tau)}{\partial y}=0,\tag{1.4}$$

$$\frac{\partial u(0,y,\tau)}{\partial x}=\frac{\partial u(x,0,\tau)}{\partial y}=0,\tag{1.5}$$

$$\frac{\partial t(h, y, \tau)}{\partial x} = \psi_1(y, \tau), \qquad \frac{\partial t(x, d, \tau)}{\partial y} = \chi_1(x, \tau), \tag{1.6}$$

$$\frac{\partial u(h, y, \tau)}{\partial x} = \psi_2(y, \tau), \qquad \frac{\partial u(x, d, \tau)}{\partial y} = \chi_2(x, \tau). \tag{1.7}$$

In order to solve this problem we employ the double Fourier transformation, applied in the region Γ.

$$(-h \leqslant x \leqslant h), \quad (-d \leqslant y \leqslant d):$$

$$F\{\varphi(x, y)\} = \iint\limits_{(\Gamma)} \exp\left(im\frac{\pi}{h}\xi - in\frac{\pi}{d}\eta\right)\varphi(\xi, \eta)\, d\xi\, d\eta = \bar{\varphi}(m, n). \tag{1.8}$$

The function $\varphi(x, y)$, is reduced by integration in the region Γ to the transform $\bar{\varphi}(m, n)$, which is a function of the two integer arguments m and n.

The inversion is obtained as follows:

$$\varphi(x, y) = \frac{1}{4hd}\sum_{m,n}^{-\infty,\,\infty} \exp\left(im\frac{\pi}{h}x + in\frac{\pi}{d}y\right)\bar{\varphi}(m, n) =$$

$$= F^{-1}\{\bar{\varphi}(m, n)\}. \tag{1.9}$$

When the function $\varphi_1(x, y)$ is integrable in the rectangle $(-2h \leqslant x \leqslant 2d - 2d \leqslant y \leqslant 2d)$, and the function $\varphi_2(x, y)$ is integrable in the region Γ, then the convolution of these functions is equal to

$$\varphi_1 * \varphi_2(x, y) = \iint\limits_{(\Gamma)} \varphi_1(x - \xi, y - \eta)\varphi_2(\xi, \eta)\, d\xi,\, d\eta. \tag{1.10}$$

If the function $\varphi_1(x, y)$ is a periodic function with period 2h in the first argument, and with period 2d in the second, then the basic law of convolution applies:

$$F\{\varphi_1 * \varphi_2\} = F\{\varphi_1\} F\{\varphi_2\}. \tag{1.11}$$

It is convenient to continue the functions t and u into the intervals $(-d \leqslant y \leqslant d)$ and $(-h \leqslant x \leqslant +h)$ as even functions, and their partial derivatives t'_x, t'_y, u'_x and u_y as odd functions. This can be accomplished on the basis of conditions (1.4) and (1.5). Then on the assumption that the order of Fourier transformation and time differentiation can be interchanged, we get a system of two ordinary differential equations:

$$\frac{d\bar{t}(m, n, \tau)}{d\tau} = -a_q \pi^2 \left(\frac{m^2}{h^2} + \frac{n^2}{d^2} \right) \bar{t}(m, n, \tau) +$$

$$+ 2a_q \left\{ (-1)^m \int_{-d}^{d} \exp\left(-in \frac{\pi}{d} \eta \right) \psi_1(\eta, \tau)\, d\eta + \right.$$

$$\left. + (-1)^n \int_{-h} \exp\left(-im \frac{\pi}{n} \xi \right) \chi_1(\xi, \tau)\, d\xi \right\} + \frac{\varepsilon\rho}{c_q} \frac{d\bar{u}(m, n, \tau)}{d\tau}, \tag{1.12}$$

$$\frac{d\bar{u}(m, n, \tau)}{d\tau} = -a_m \pi^2 \left(\frac{m^2}{h^2} + \frac{n^2}{d^2} \right) \bar{u}(m, n, \tau) - a_m \delta \pi^2 \left(\frac{m^2}{h^2} + \frac{n^2}{d^2} \right) \times$$

$$\times \bar{t}(m, n, \tau) + 2a_m \left\{ (-1)^m \int_{-d}^{d} \exp\left(-in \frac{\pi}{d} \eta \right) \psi_2(\eta, \tau)\, d\eta + \right.$$

$$\left. + (-1)^n \int_{-h}^{h} \exp\left(-im \frac{\pi}{h} \xi \right) \chi_2(\xi, \tau)\, d\xi \right\} + 2 \frac{\varepsilon\rho}{c_q} \times$$

$$\times \left\{ (-1)^m \int_{-d}^{d} \exp\left(-in \frac{\pi}{d} \eta \right) \psi_1(\eta, \tau)\, d\eta + (-1)^n \int_{-h}^{h} \times \right.$$

$$\left. \times \exp\left(-im \frac{\pi}{h} \xi \right) \chi_1(\xi, \tau)\, d\xi \right\} \tag{1.13}$$

with initial conditions

$$\bar{t}(m, n, 0) = \bar{f}(m, n), \quad \bar{u}(m, n, 0) = \bar{\varphi}(m, n). \tag{1.14}$$

The system of equations (1.12) and (1.13) can be solved by use of the Laplace transformation. In this way we get the following solution for $\bar{t}(m, n, \tau)$ and $\bar{u}(m, n, \tau)$:

$$\bar{t}(m, n, \tau) = \sum_{j=1}^{2} (-1)^{j+1} \left\{ A_j \bar{\varphi}(m, n) \exp\left[-\nu_j^2 a_q \mathrm{Lu}\, \pi^2 \times \right. \right.$$

$$\left. \times \left(\frac{m^2}{h^2} + \frac{h^2}{d^2} \right) \tau \right] + B_j \bar{f}(m, n) \exp\left[-a_q \mathrm{Lu}\, \nu_j^2 \pi^2 \times \right.$$

$$\times\left(\frac{m^2}{h^2}+\frac{n^2}{d^2}\right)\tau\,\Big] + C_j \exp\left[-a_q\,\mathrm{Lu}\,\nu_j^2\pi^2\left(\frac{m^2}{h^2}+\frac{n^2}{d^2}\right)\tau\right]\overset{\tau}{*}\times$$

$$\times\left[(-1)^m\int_{-d}^{d}\exp\left(in\frac{\pi}{d}\eta\right)\psi_1(\eta,\tau)\,d\eta+\right.$$

$$+(-1)^n\int_{-h}^{h}\exp\left(-im\frac{\pi}{h}\xi\right)\chi_1(\xi,\tau)\,d\xi + D_j\times$$

$$\times\exp\left[-a_q\,\mathrm{Lu}\,\nu_j^2\pi^2\left(\frac{m^2}{h^2}+\frac{n^2}{d^2}\right)\tau\right]\overset{\tau}{*}\left[(-1)^m\int_{-d}^{d}\times\right.$$

$$\times\exp\left(-in\frac{\pi}{d}\eta\right)\psi_2(\eta,\tau)\,d\eta+(-1)^n\int_{-h}^{h}\times$$

$$\left.\left.\times\exp\left(-im\frac{\pi}{h}\xi\right)\chi_2(\xi,\tau)\,d\xi\right]\right\}, \qquad (1.15)$$

here $\overset{\tau}{*}$ denotes convolution with respect to τ from 0 to τ; and ν_j^2 determined by equations (1.9) and (1.10).

The constant coefficients are equal to

$$A_j=\frac{\varepsilon\rho}{c_q(\nu_1^2-\nu_2^2)},\qquad B_j=\frac{\nu_j^2-1}{\nu_1^2-\nu_2^2},$$

$$C_j=-\frac{2a_q\left[\nu_j^2(1+\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\,\mathrm{Lu})-1\right]}{\nu_1^2-\nu_2^2},$$

$$D_j=\frac{2a_q\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}\,\nu_j^2}{\delta(\nu_1^2-\nu_2^2)}. \qquad (1.16)$$

The transform $\bar{u}(m,n,\tau)$ is also given by the right-hand side equation (1.15), when the coefficients A_j, B_j, C_j, D_j are replaced the corresponding coefficients A_j', B_j', C_j' and D_j. The latter are ven by

$$A_j'=\frac{\nu_j^2-\dfrac{1}{\mathrm{Lu}}-\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}}{\nu_1^2-\nu_2^2},\qquad B_j'=\frac{\delta}{\nu_1^2-\nu_2^2},$$

$$C_j' = \frac{2a_q \delta \operatorname{Lu} \nu_j^2}{\nu_1^2 - \nu_2^2}, \qquad D_j' = \frac{2a_q(\operatorname{Lu} \nu_j^2 - 1)}{\nu_1^2 - \nu_2^2}. \tag{1.17}$$

Before obtaining the inversion it is necessary to note certain properties of the theta function

$$\vartheta_3(v, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n}, \qquad (z = e^{i\pi v}, \quad q = e^{-i\pi\tau}). \tag{1.18}$$

The function $\bar{K}(m, n, z)$, defined by the following equation:

$$\bar{K}(m, n, z) = \exp\left[-\pi^2\left(\frac{m^2}{h^2} + \frac{n^2}{d^2}\right) z\right], \tag{1.19}$$

has the following inversion:

$$K(x, y, z) = \frac{1}{4hd} \vartheta_3\left[\frac{x}{2h}, \frac{i\pi z}{h^2}\right] \vartheta_3\left[\frac{y}{2d}, \frac{i\pi z}{d^2}\right]. \tag{1.20}$$

From known properties of theta functions, it follows that the function K(x, y, z) is periodic in x with period 2h and in y with period 2d. For this reason, and by virtue of equations (1.19) and (1.20), it is easy to obtain the inversions with respect to m and n.

After the final transformations we have

$$t(x, y, \tau) = \frac{1}{4hd} \sum_{j=1}^{2} (-1)^{j+1} \Bigg\{ A_j \iint\limits_{(\Gamma)} \vartheta_3\left(\frac{x-\xi}{2h}, i \operatorname{Lu}, \nu_j^2 \pi \operatorname{Fo}_1\right) \times$$

$$\times \vartheta_3\left(\frac{y-\eta}{2d}, i \operatorname{Lu} \pi \nu_j^2 \operatorname{Fo}_2\right) \varphi(\xi, \eta)\, d\xi, d\eta +$$

$$+ B_j \iint\limits_{(\Gamma)} \vartheta_3\left(\frac{x-\xi}{2h}, i\pi \nu_j^2 \operatorname{Lu} \operatorname{Fo}_1\right) \vartheta_3\left(\frac{y-\eta}{2d}, i\pi \nu_j^2 \operatorname{Lu} \operatorname{Fo}_2\right) \times$$

$$\times f(\xi, \eta)\, d\xi\, d\eta + C_j \int_0^{\tau} d\theta \Bigg(\vartheta_3\left[\frac{h-x}{2h}, i\pi \nu_j^2 \operatorname{Lu} \operatorname{Fo}_1 \left(1 - \frac{\theta}{\tau}\right)\right] \times$$

$$\times \int_{-d}^{d} \vartheta_3\left[\frac{y-\eta}{2d}, i\pi \nu_j^2 \operatorname{Lu} \operatorname{Fo}_2 \left(1 - \frac{\theta}{\tau}\right)\right] \psi_1(\eta, \theta)\, d\eta +$$

$$+ \vartheta_3 \Bigg[\frac{d-y}{2d}, i\pi \nu_j^2 \mathrm{Lu\,Fo}_2 \Bigg(1 - \frac{\theta}{\tau}\Bigg)\Bigg] \int\limits_{-h}^{h} \vartheta_3 \times$$

$$\times \Bigg[\frac{x-\xi}{2\lambda}, i\pi \nu_j^2 \mathrm{Lu\,Fo}_1 \Bigg(1 - \frac{\vartheta}{\theta}\Bigg)\Bigg] \chi_1(\xi, \theta)\, d\xi \Bigg) +$$

$$+ D_j \int\limits_{0}^{\tau} d\theta \Bigg(\vartheta_3 \Bigg[\frac{h-x}{2h}, i\pi \nu_j^2 \mathrm{Lu\,Fo} \Bigg(1 - \frac{\theta}{\tau}\Bigg)\Bigg] \int\limits_{-d}^{d} \vartheta_3 \times$$

$$\times \Bigg[\frac{y-\eta}{2d}, i\pi \nu_j^2 \mathrm{Lu\,Fo}_2 \Bigg(1 - \frac{\vartheta}{\tau}\Bigg)\Bigg] \psi_2(\eta, \vartheta)\, d\eta + \vartheta_3 \times$$

$$\times \Bigg[\frac{d-y}{2d}, i\pi \nu_j^2 \mathrm{Lu\,Fo}_2 \Bigg(1 - \frac{\tau}{\theta}\Bigg)\Bigg] \int\limits_{-h}^{h} \vartheta_3 \Bigg[\frac{x-\xi}{2h}, i\pi \nu_j^2 \mathrm{LuFo}_1 \times$$

$$\times \Bigg(1 - \frac{\tau}{\theta}\Bigg)\Bigg] \chi_2(\xi, \tau)\, d\xi \Bigg)\Bigg\}. \tag{1.21}$$

The inversion $u(x, y, \tau)$ is obtained by the same method; there results an expression for $u(x, y, \tau)$, which is equation (1.21) with the coefficients A_j, B_j, C_j, and D_j replaced by A_j', B_j', C_j' and D_j'.

2. TWO-DIMENSIONAL PLATE. BOUNDARY CONDITIONS OF THE FIRST AND THIRD KIND.

By means of simultaneous application of two-dimensional, finite, complex Fourier transformations and Laplace transformations the following problems can also be solved. These problems relate to non-isothermal mass transfer in a plate with finite dimensions (h × d) and with boundary conditions of the first and third kind.

Problem 1. Solve the system of differential equations (1.1) and (1.2) (see Section 1) with the following initial conditions:

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and boundary conditions

$$\frac{\partial t(0, y, \tau)}{\partial x} = \frac{\partial t(x, 0, \tau)}{\partial y} = 0, \tag{4}$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = \frac{\partial u(x, 0, \tau)}{\partial y} = 0, \tag{5}$$

$$-\lambda_q \frac{\partial t(h, y, \tau)}{\partial x} + \psi_1(y, \tau) = 0, \tag{6}$$

$$a_m \frac{\partial u(h, y, \tau)}{\partial x} + a_m \delta \frac{\partial t(h, y, \tau)}{\partial x} + \psi_2(y, \tau) = 0, \tag{7}$$

$$-\lambda_q \frac{\partial t(x, d, \tau)}{\partial y} + \chi_1(x, \tau) = 0, \tag{8}$$

$$a_m \frac{\partial u(x, d, \tau)}{\partial y} + a_m \delta \frac{\partial t(x, d, \tau)}{\partial y} + \chi_2(x, \tau) = 0. \tag{9}$$

In order to solve equations (1)-(9) we employ the two-dimensional, finite, complex Fourier transformation in the region $\Gamma\,(-h \leqslant x \leqslant +h - d \leqslant y \leqslant +d)$

$$\bar{\varphi}(m, n) = \int\int \exp\left(-im\frac{\pi}{h}\xi - in\frac{\pi}{d}\eta\right)\varphi(\xi, \eta)\,d\xi\,d\eta \tag{10}$$

and the corresponding inversion formula

$$\varphi(x, y) = \frac{1}{4hd} \sum_{(m, n)}^{-\infty, +\infty} \exp\left(im\frac{\pi}{h}x + in\frac{\pi}{d}y\right)\bar{\varphi}(m, n). \tag{11}$$

If

$$\begin{aligned} &\bar{\varphi}_1(m, n) \div \varphi_1(x, y), \\ &\bar{\varphi}_2(m, n) \div \varphi_2(x, y), \end{aligned} \tag{12}$$

then with the condition that

$$\begin{aligned} &\varphi_1(x + 2h, y) = \varphi_1(x, y), \\ &\varphi_1(x, y + 2d) = \varphi_1(x, y) \end{aligned} \tag{13}$$

the convolution theorem can be written as

$$\bar{\varphi}_1(m, n)\,\bar{\varphi}_2(m, n) \div \iint_{(\Gamma)} \varphi_1(x - \xi, y - \eta)\,\varphi_2(\xi, \eta)\,d\xi\,d\eta. \tag{14}$$

It is assumed here that $\varphi_2(x, y)$ can be integrated in the region Γ, and $\varphi_1(x, y)$ can be integrated in a wider region $(-2h \leqslant x, \leqslant 2h, \; -2d \leqslant y \leqslant +2d)$.

Replacing φ with partial derivatives and integrating equation (10) by parts, we get formulas with the transforms of the partial derivatives:

$$\frac{\partial \varphi}{\partial x} \doteqdot im\frac{\pi}{h}\bar{\varphi}(m, n)+(-1)^m\int\limits_{-d}^{+d}\exp\left(-in\frac{\pi}{d}\eta\right)\times$$

$$\times\left[\varphi(h, \eta)+\varphi(-h, \eta)\right]d\eta, \tag{15}$$

$$\frac{\partial \varphi}{\partial y} \doteqdot in\frac{\pi}{d}\bar{\varphi}(m, n)+(-1)^n\int\limits_{-h}^{+h}\exp\left(-im\frac{\pi}{d}\eta\right)\times$$

$$\times\left[\varphi(\xi, d)+\varphi(\xi, -d)\right]d\xi, \tag{16}$$

$$\frac{\partial^2 \varphi}{\partial x^2} \doteqdot -m^2\frac{\pi^2}{h^2}\bar{\varphi}(m, n)+(-1)^m\int\limits_{-d}^{+d}\exp\left(-in\frac{\pi}{d}\eta\right)\times$$

$$\times\left\{\frac{\partial \varphi(h, \eta)}{\partial \xi}-\frac{\partial \varphi(-h, \eta)}{\partial \xi}+\right.$$

$$\left.+im\frac{\pi}{h}\left[\varphi(h, \eta)-\varphi(-h, \eta)\right]\right\}d\eta, \tag{17}$$

$$\frac{\partial^2 \varphi}{\partial y^2} \doteqdot -n^2\frac{\pi^2}{h^2}\bar{\varphi}(m, n)+(-1)^n\int\limits_{-h}^{+h}\exp\left(-im\frac{\pi}{h}\xi\right)\times$$

$$\times\left\{\frac{\partial \varphi(\xi, d)}{\partial \eta}-\frac{\partial \varphi(\xi, -d)}{\partial \eta}+\right.$$

$$\left.+in\frac{\pi}{d}\left[\varphi(\xi, d)-\varphi(\xi, -d)\right]\right\}d\xi. \tag{18}$$

If we now continue into the intervals ($-h \leqslant x \leqslant 0$) and ($-d \leqslant y \leqslant 0$) the functions t and u as even functions, and the partial derivatives $\frac{\partial t}{\partial x}$, $\frac{\partial t}{\partial y}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ as odd functions, in virtue of equations (4) and (5), and if we take the transforms of equations (1) to (9) with respect to x and y in accordance with equations (15) to (18), then we get instead of the system of partial differential equations (1) and (2) a system of two ordinary differential equations for $\bar{t}(m, n, \tau)$ and $\bar{u}(m, n, \tau)$:

$$\left.\begin{aligned} \frac{d\bar{t}(m,n,\tau)}{d\tau} &= -a_q\sigma\,\bar{t}(m,n,\tau) + \frac{2a_q}{\lambda_q}S_1(\tau) + \\ &\quad + \frac{\varepsilon\rho}{c}\,\frac{d\bar{u}(m,n,\tau)}{d\tau} \\ \frac{d\bar{u}(m,n,\tau)}{d\tau} &= -a_m\sigma\,\bar{u}(m,n,\tau) - 2S_2(\tau) - \\ &\quad - a_m\delta\sigma\,\bar{t}(m,n,\tau) \end{aligned}\right\}, \tag{19}$$

where

$$\sigma = \pi^2\left(\frac{m^2}{h^2} + \frac{h^2}{d^2}\right), \tag{20}$$

$$\begin{aligned} S_j(\tau) &= (-1)^m\int_{-d}^{+d}\exp\left(-in\frac{\pi}{d}\eta\right)\psi_j(\eta,\tau)\,d\eta + \\ &\quad + (-1)^n\int_{-h}^{+h}\exp\left(-im\frac{\pi}{h}\xi\right)\chi_j(\xi,\tau)\,d\xi. \end{aligned} \tag{21}$$

The transformed initial conditions (3) are given by the following:

$$\bar{t}(m,n,0) = \bar{f}_1(m,n), \quad \bar{u}(m,n,0) = \bar{f}_2(m,n). \tag{22}$$

Applying the Laplace transform with respect to τ to equation (19) we get a system of two algebraic equations of the first degree for two unknown functions $\bar{t}(m,n,s)$ and $\bar{u}(m,n,s)$. Solving this system and inverting with respect to s, we get

$$\begin{aligned} t(m,n,\tau) = \sum_{i=1}^{2}\Big\{ &\exp\left(-a_q\,\mathrm{Lu}\,\nu_i^2\sigma\tau\right)\Big[A_i\bar{f}_1(m,n) + \\ &+ B_i\bar{f}_2(m,n)\Big] + \exp(-a_q\,\mathrm{Lu}\,\nu_i^2\sigma\tau)\times \\ &\times\left[D_i\overset{\tau}{*}S_2(\tau) + C_i\overset{\tau}{*}S_1(\tau)\right]\Big\}. \end{aligned} \tag{23}$$

Here the symbol $\overset{\tau}{*}$ denotes convolution with respect to τ from 0 to τ. The coefficients A_i, B_i, C_i, D_i are determined by the following formulas:

$$A_l = (-1)^{l+1}\frac{\nu_l^2 - 1}{\nu_1^2 - \nu_2^2}, \quad B_l = (-1)^{l+1}\frac{\varepsilon \mathrm{Ko}\,\mathrm{Pn}}{\nu_1^2 - \nu_2^2},$$

$$C_l = \frac{2}{c_q \gamma_0} A_l, \quad D_l = 2\nu_l^2 B_l. \tag{24}$$

The second function $\overline{u}(m, n, \tau)$ is also given by the right-hand side of equation (23), when the coefficients A_l, B_l, C_l, D_l are replaced by the coefficients A_l, B_l, C_l, D_l. The latter are given by

$$A_l' = (-1)^{l+1}\frac{\delta}{\nu_1^2 - \nu_2^2},$$

$$B_l' = \frac{(-1)^{l+1}}{\nu_1^2 - \nu_2^2}\left(\nu_l^2 - \varepsilon \mathrm{Ko}\,\mathrm{Pn} - \frac{1}{\mathrm{Lu}}\right),$$

$$C_l' = (-1)^{l+1}\frac{2\delta}{c_q \gamma_0 (\nu_1^2 - \nu_2^2)},$$

$$D_l' = (-1)^{l+1}\frac{\left(\nu_l^2 - \frac{1}{\mathrm{Lu}}\right)}{\nu_1^2 - \nu_2^2}. \tag{25}$$

In order to obtain the final solution of the problem we have to invert equation (23) with respect to m and n. This can be done by means of the convolution theorem and use of the theta functions. We recall that

$$\vartheta_3(v, \tau) = \sum_{n=-\infty}^{+\infty} q^{n^2} z^{2n} \left(z = e^{l\pi v}, q = e^{l\pi\tau}\right). \tag{26}$$

Consequently, in accordance with equations (11), (20) and (26),

$$e^{-\sigma\varsigma} \doteqdot \frac{1}{4hd}\,\vartheta_3\left(\frac{x}{2h}, \frac{i\pi\varsigma}{h^2}\right)\vartheta_3\left(\frac{y}{2d}, \frac{i\pi\varsigma}{d^2}\right). \tag{27}$$

In the right-hand side of equation (27) we have a function which is periodic in x with period 2h and in y with period 2d. For this reason, and by virtue of equation (25) we obtain the inversions of the first two terms of the right-hand side of equation (23) using the convolution theorem, equation (14). The remaining terms are inverted directly by means of equation (11). As a result we get

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + \\ + C_1 K_{11} + C_2 K_{21} + D_1 K_{12} + D_2 K_{22}, \tag{28}$$

where

$$I_{mn} = \frac{1}{4hd} \iint\limits_{(\Gamma)} \vartheta_3 \left(\frac{x-\xi}{2h}, \frac{i\pi \mathrm{Lu}\, a_q \nu_l^2 \tau}{h^2} \right) \times \\ \times \vartheta_3 \left(\frac{y-h}{2d}, \frac{i\pi \mathrm{Lu}\, a_q \nu_m^2 \tau}{d^2} \right) f_n(\xi, \eta)\, d\xi\, d\eta. \tag{29}$$

$$K_{mn} = \frac{1}{4hd} \int\limits_0^\tau d\theta \left\{ \vartheta_3 \left(\frac{h-x}{2h}, \frac{i\pi \mathrm{Lu}\, a_q \nu_m^2 (\tau-\theta)}{h^2} \right) \times \right. \\ \times \int\limits_{-d}^{+d} \vartheta_3 \left(\frac{y-h}{2d}, \frac{i\pi a_q \mathrm{Lu}\, \nu_m^2 (\tau-\theta)}{d^2} \right) \psi_n(\eta, \theta)\, d\eta + \\ + \vartheta_3 \left(\frac{d-y}{2d}, \frac{i\pi \mathrm{Lu}\, a_q \nu_m^2 (\tau-\theta)}{d^2} \right) \times \\ \left. \times \int\limits_{-c}^{+c} \vartheta_3 \left(\frac{x-\xi}{2h}, \frac{i\pi \mathrm{Lu}\, a_q \nu_m^2 (\tau-\theta)}{h^2} \right) \chi_n(\xi, \theta)\, d\xi \right\}. \tag{30}$$

By introducing the following notation,

$$A_1 = A_{11},\ A_2 = A_{21},\ B_1 = A_{12},\ B_2 = A_{22},\ C_1 = B_{11}, \\ C_2 = B_{21},\ D_1 = B_{12},\ D_2 = B_{22},$$

equation (28) can be rewritten in the following form:

$$t(x, y, \tau) = \sum_{i,j} (A_{ij} I_{ij} + B_{ij} \mathrm{K}_{ij}), \\ i, j = 1,2.$$

The inversion of $\bar{u}(m, n, \tau)$ has the following form:

$$u(x, y, \tau) = \sum_{(i,j)} (A'_{ij} I_{ij} + B'_{ij} K_{ij}) (i, j = 1,2). \tag{32}$$

Problem 2. Solve the system of differential equations (1.1) and (1.2) with the following initial conditions:

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and boundary conditions

$$\frac{\partial t(0, y, \tau)}{\partial x} = \frac{\partial t(x, 0, \tau)}{\partial y} = 0, \tag{4}$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = \frac{\partial u(x, 0, \tau)}{\partial y} = 0, \tag{5}$$

$$t(h, y, \tau) = \psi_1(y, \tau), \tag{6}$$

$$u(h, y, \tau) = \psi_2(y, \tau), \tag{7}$$

$$-\lambda_q \frac{\partial t(x, d, \tau)}{\partial y} + \chi_1(x, \tau) = 0, \tag{8}$$

$$a_m \left(\frac{\partial u(x, d, \tau)}{\partial y} + \delta \frac{\partial t(x, d, \tau)}{\partial y} \right) + \chi_2(x, \tau) = 0. \tag{9}$$

We employ here the two-dimensional, finite, complex Fourier transformation in the region $\Gamma(-h \leqslant x \leqslant +h,\ d \leqslant y \leqslant +d)$:

$$\bar{\varphi}(m, n) = \iint_{(\Gamma)} \exp\left[-i\left(m - \frac{1}{2} \right) \frac{\pi}{h} \xi - \right.$$
$$\left. - in \frac{\pi}{d} \eta \right] \varphi(\xi, \eta)\, d\xi\, d\eta, \tag{10}$$

$$\varphi(x, y) = \frac{1}{4hd} \sum_{(m, n)}^{-\infty, +\infty} \exp\left[i\left(m - \frac{1}{2} \right) \frac{\pi}{h} x + \right.$$
$$\left. + in \frac{\pi}{d} y \right] \bar{\varphi}(m, n). \tag{11}$$

When applying transformation (10) to equations (1.1) and (1.2) we get a system of two ordinary differential equations for the functions $\bar{t}(m, n, \tau)$, $\bar{u}(m, n, \tau)$:

$$\left.\begin{aligned} \frac{d\bar{t}(m,n,\tau)}{d\tau} &= -a_q\sigma\bar{t}(m,n,\tau) + 2a_qS_1(\tau) + \frac{2a_q}{\lambda_q}I_1(\tau) + \\ &\quad + \frac{\varepsilon\rho}{c}\frac{d\bar{u}(m,n,\tau)}{d\tau} \\ \frac{d\bar{u}(m,n,\tau)}{d\tau} &= -a_m\sigma\bar{u}(m,n,\tau) + \\ &+ 2a_mS_2(\tau) - 2J_2(\tau) - a_m\delta\sigma\bar{u}(m,n,\tau) + 2a_m\delta S_1(\tau) \end{aligned}\right\}, \tag{12}$$

where

$$\sigma = \pi^2\left[\frac{\left(m-\frac{1}{2}\right)^2}{h^2} + \frac{n^2}{d^2}\right], \tag{13}$$

$$\left.\begin{aligned} S_j(\tau) &= (-1)^m i\int_{-d}^{+d}\exp\left(-in\frac{\pi}{d}\eta\right)\times \\ &\times\left[i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\right]\psi_j(\eta,\tau)\,d\eta \\ I_j(\tau) &= (-1)^n\int_{-h}^{+h}\exp\left[-i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\xi\right]\chi_j(\xi,\tau)\,d\xi \end{aligned}\right\}. \tag{14}$$

Solving this system by means of Laplace transformation with respect to time, we get

$$\begin{aligned} \bar{t}(m,n,\tau) &= \sum_{i=1}^{2}\{\exp(-a_q\mathrm{Lu}\,\nu_i^2\sigma\tau)\,[A_i\bar{f}_1(m,n) + \\ &+ B_i\bar{f}_2(m,n) + \overset{\tau}{C_i^*}\,I_1(\tau) + \overset{\tau}{D_i^*}\,I_2(\tau) + a_m\overset{\tau}{D_i^*}\,S_2(\tau) + \\ &+ \overset{\tau}{E_i^*}\,S_1(\tau)]\}. \end{aligned} \tag{15}$$

The coefficients A_i, B_i, C_i, D_i are determined by equations (25) of the preceding problem and the coefficients D_i by the following:

$$E_i = (-1)^{i+1}\frac{2a_q}{\nu_1^2-\nu_2^2}\,[\nu_i^2(1+\varepsilon\,\mathrm{LuKoPn}) - 1]. \tag{16}$$

The second function, $\bar{u}(m, n, \tau)$, is obtained from equation (15) by replacing the coefficients A_l, B_l, C_l, D_l, E_l, with A'_l, B'_l, C'_l, D'_l and E'_l, the last being given by

$$E'_l = 2a_m \delta (-1)^{l+1} \frac{\nu_1^2}{\nu_1^2 - \nu_2^2}. \tag{17}$$

Using the convolution theorem and equation (11) we finally get

$$\begin{gathered} t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + E_1 K_{11} + \\ + E_2 K_{21} + a_m D_1 K_{12} + a_m D_2 K_{22} + C_1 M_{11} + C_2 M_{21} + \\ + D_1 M_{12} + D_2 M_{22}, \end{gathered} \tag{18}$$

where

$$\begin{gathered} I_{kl} = \frac{1}{4hd} \iint_{(\Gamma)} \vartheta_2 \left(\frac{x-\xi}{2h}, \frac{\alpha_k i \pi \tau}{h^2} \right) \vartheta_3 \left(\frac{y-\eta}{2d}, \frac{\alpha_k i \pi \tau}{d^2} \right) \times \\ \times f_l(\xi, \eta)\, d\xi\, d\eta, \end{gathered} \tag{19}$$

$$\begin{gathered} K_{kl} = \frac{1}{4hd} \int_0^\tau d\theta \left\{ \frac{\partial}{\partial x} \vartheta_2 \left[\frac{h-x}{2h}, \frac{\alpha_k i \pi (\tau - \theta)}{h^2} \right] \times \right. \\ \left. \times \int_{-d}^{+d} \vartheta_3 \left[\frac{y-\eta}{2d}, \frac{\alpha_k i \pi (\tau - \theta)}{d^2} \right] \psi_l(\eta, d)\, d_\eta \right\}, \end{gathered} \tag{20}$$

$$\begin{gathered} M_{kl} = \frac{1}{4hd} \int_0^\tau d\theta \left\{ \vartheta_3 \left[\frac{d-y}{2d}, \frac{\alpha_k i \pi (\tau - \theta)}{d^2} \right] \times \right. \\ \left. \times \int_{-h}^{+h} \vartheta_2 \left[\frac{x-\xi}{2h}, \frac{\alpha_k i \pi (\tau - \theta)}{h^2} \right] \chi_l(\xi\, d)\, d\xi \right\}, \end{gathered} \tag{21}$$

$$\alpha_k = \nu_k^2 \mathrm{Lu} a_q, \quad k, l = 1, 2.$$

The inversion of $\bar{u}(m, n, \tau)$ is given in equation (18) with $A_l \ldots$ replaced by $A'_l \ldots$

Problem 3. Solve the system of differential equation (1.1) and (1.2) with initial conditions:

$$t(x, y, 0)=f_1(x, y), \quad u(x, y, 0)=f_2(x, y) \tag{3}$$

and boundary conditions

$$\frac{\partial t(0, y, \tau)}{\partial x}=\frac{\partial t(x, 0, \tau)}{\partial y}=0, \tag{4}$$

$$\frac{\partial u(0, y, \tau)}{\partial x}=\frac{\partial u(x, 0, \tau)}{\partial y}=0, \tag{5}$$

$$t(h, y, \tau)=\psi_1(y, \tau), \tag{6}$$

$$u(h, y, \tau)=\psi_2(y, \tau), \tag{7}$$

$$t(x, d, \tau)=\chi_1(x, \tau), \tag{8}$$

$$u(x, d, \tau)=\chi_2(x, \tau). \tag{9}$$

In the solution of this problem too we employ the two-dimensional finite, complex Fourier transformation in the region $\Gamma(-h \leqslant x \leqslant +h,\ -d \leqslant y \leqslant +d)$:

$$\bar{\varphi}(m, n)=\iint_{(\Gamma)} \exp\left[-i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\xi - i\left(n-\frac{1}{2}\right)\frac{\pi}{d}\eta\right]\varphi(\xi, \eta)\,d\xi\,d\eta \tag{10}$$

and the corresponding formula

$$\varphi(x, y)=\frac{1}{4hd}\sum_{(m, n)}^{-\infty,+\infty} \exp\left[i\left(m-\frac{1}{2}\right)\frac{\pi}{h}x + i\left(n-\frac{1}{2}\right)\frac{\pi}{d}y\right]\bar{\varphi}(m, n). \tag{11}$$

If,

$$\begin{aligned}\bar{\varphi}_1(m, n) &\div \varphi_1(x, y),\\ \bar{\varphi}_2(m, n) &\div \varphi_2(x, y),\end{aligned} \tag{12}$$

then with the following condition:

$$\begin{aligned}\varphi_1(x+2h, y) &= -\varphi_1(x, y),\\ \varphi_1(x, y+2d) &= -\varphi_1(x, y)\end{aligned} \tag{13}$$

we can apply the convolution theorem

$$\bar{\varphi}_1(m, n)\,\bar{\varphi}_2(m, n) \doteqdot \iint\limits_{(\Gamma)} \varphi_1(x-\xi, y-\eta)\,\varphi_2(\xi, \eta)\,d\xi\,d\eta. \tag{14}$$

Substituting partial derivatives into equation (10) and integrating by parts we obtain a system of two ordinary differential equations for $\bar{t}(m, n, \tau)$ and $\bar{u}(m, n, \tau)$:

$$\left.\begin{aligned} \frac{d\bar{t}(m,n,\tau)}{d\tau} &= -a_q\,\sigma\bar{t}(m,n,\tau) + a_q S_1(\tau) + \frac{\varepsilon\rho}{c}\,\frac{d\bar{u}(m,n,\tau)}{d\tau} \\ \frac{d\bar{u}(m,n,\tau)}{d\tau} &= -a_m\,\sigma\bar{u}(m,n,\tau) + a_m S_2(\tau) - a_m\,\delta\sigma\bar{t}(m,n,\tau) + \\ &\quad + a_m S_1(\tau), \end{aligned}\right\}, \tag{15}$$

where,

$$\sigma = \frac{\left(m-\frac{1}{2}\right)^2}{h^2} + \frac{\left(n-\frac{1}{2}\right)^2}{d^2}, \tag{16}$$

$$\begin{aligned} S_j(\tau) = 2(-1)^m i\int\limits_{-d}^{+d} \exp\left[-i\left(n-\frac{1}{2}\right)\frac{\pi}{d}\eta\right]\left[i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\right]\times \\ \times\psi_1(\eta,\tau)\,d\eta + 2(-1)^n\, i\int\limits_{-h}^{+h} \exp\left[-i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\xi\right]\times \\ \times\left[i\left(n-\frac{1}{2}\right)\frac{\pi}{d}\right]\chi_1(\xi,\tau)\,d\xi. \end{aligned} \tag{17}$$

The final solution of equation (15) is

$$\begin{aligned} t(x,y,\tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + 2E_1 K_{11} + \\ + 2E_2 K_{21} + a_m D_1 K_{12} + a_m D_2 K_{22}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} I_{kl} = \frac{1}{4hd}\iint\limits_{(\Gamma)} \vartheta_2\left(\frac{x-\xi}{2h}, \frac{i\pi\alpha_k\tau}{h^2}\right)\vartheta_2\left(\frac{y-\eta}{2d}, \frac{i\pi\alpha_k\tau}{d^2}\right)\times \\ \times f_l(\xi,\eta)\,d\xi,\,d\eta, \end{aligned} \tag{19}$$

$$K_{kl} = \frac{1}{2hd}\int_0^\tau d\theta \left\{ \frac{\partial}{\partial x}\vartheta_2\left[\frac{h-x}{2h}, \frac{i\pi\alpha_k(\tau-\theta)}{h^2}\right]\times\right.$$

$$\left.\times\int_{-d}^{+d}\vartheta_2\left[\frac{y-\eta}{2d}, \frac{i\pi\alpha_k(\tau-\theta)}{d^2}\right]\psi_l(\eta,\theta)\,d\eta\right\}+$$

$$+\frac{1}{2hd}\int_0^\tau d\theta \left\{ \frac{\partial}{\partial x}\vartheta_2\left[\frac{d-y}{2d}, \frac{i\pi\alpha_k(\tau-\theta)}{d^2}\right]\times\right.$$

$$\left.\times\int_{-h}^{+h}\vartheta_2\left[\frac{x-\xi}{2h}, \frac{i\pi\alpha_k(\tau-\theta)}{h^2}\right]\chi_l(\xi,\alpha)\,d\xi\right\}, \tag{20}$$

$$\alpha_k = a_q \,\mathrm{Lu}\, \nu_k^2.$$

The inversion of $\bar{u}(m, n, \tau)$ is given by equation (18) with A_l ... replaced by A'_l ....

Problem 4. Solve the system of differential equations (1.1) and (1.2) with the following initial conditions:

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and boundary conditions

$$\frac{\partial t(0, y, \tau)}{\partial x} = t(x, 0, \tau) = 0, \tag{4}$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = u(x, 0, \tau) = 0, \tag{5}$$

$$-\lambda_q \frac{\partial t(h, y, \tau)}{\partial x} + \psi_1(y, \tau) = 0, \tag{6}$$

$$a_m\left[\frac{\partial u(h, y, \tau)}{\partial x} + \delta\frac{\partial t(h, y, \tau)}{\partial x}\right] + \psi_2(y, \tau) = 0, \tag{7}$$

$$t(x, d, \tau) = \chi_1(x, \tau), \tag{8}$$

$$u(x, d, \tau) = \chi_2(x, \tau). \tag{9}$$

Here, again, we use the two-dimensional, finite, complex Fourier transformation in the region $\Gamma(-h \leqslant x \leqslant +h, -d \leqslant y \leqslant +d)$:

$$\bar{\varphi}(m,\ n) = \iint\limits_{(\Gamma)} \exp\left(-im\frac{\pi}{h}\xi - in\frac{\pi}{d}\eta\right)\varphi(\xi, \eta)\,d\xi\,d\eta \tag{10}$$

and the corresponding inversion formula

$$\varphi(x, y) = \frac{1}{4hd}\sum_{(m,n)}^{+\infty,\,-\infty} \exp\left(im\frac{\pi}{h}x + in\frac{\pi}{d}y\right)\bar{\varphi}(m, n). \tag{11}$$

Continuing the functions τ and u into the interval $(-h \leqslant x \leqslant 0)$ as even functions, and into the interval $(-d \leqslant y \leqslant 0)$ as odd functions, on the basis of conditions (4) and (5), we get

$$t(x, y, \tau) = t(-x. y, \tau),\quad u(x, y, \tau) = u(-x, y, \tau), \tag{12}$$

$$t(x, y, \tau) = -t(x, -y, \tau),\quad u(x, y, \tau) = -u(x, -y, \tau), \tag{13}$$

$$\frac{\partial t(x, y, \tau)}{\partial x} = -\frac{\partial t(-x, y, \tau)}{\partial x},$$

$$\frac{\partial u(x, y, \tau)}{\partial x} = -\frac{\partial u(-x, y, \tau)}{\partial x}, \tag{14}$$

$$\frac{\partial t(x, y, \tau)}{\partial y} = \frac{\partial t(x, -y, \tau)}{\partial y},$$

$$\frac{\partial u(x, y, \tau)}{\partial y} = \frac{\partial u(x, -y, \tau)}{\partial y}, \tag{15}$$

$$f_1(x,y) = f_1(-x, y) = -f_1(x,y),\quad f_2(x,y) = f_2(-x,y) = -f_2(x,y), \tag{16}$$

$$\psi_1(y, \tau) = -\psi_1(-y, \tau),\quad \psi_2(y, \tau) = -\psi_2(-y, \tau), \tag{17}$$

$$\chi_1(x, \tau) = \chi_1(-x, \tau),\quad \chi_2(x, \tau) = \chi_2(-x, \tau). \tag{18}$$

Using equation (10) to transform equations (1) to (9) with respect to x and y we obtain a system of two ordinary differential equations for, $\bar{t}(m, n, \tau)$ and $\bar{u}(m, n, \tau)$.

Finally we get,

$$
\begin{gathered}
t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + C_1 K_{11} + \\
+ C_2 K_{21} + D_1 K_{11} + D_2 R_{22} + E_1 M_{11} + E_2 M_{21} + \\
+ a_m D_1 M_{12} + a_m D_2 M_{22},
\end{gathered} \tag{19}
$$

$$
\begin{gathered}
I_{kl} = \frac{1}{4hd} \iint\limits_{(\Gamma)} \vartheta_3 \left(\frac{x-\xi}{2h}, \frac{i\pi\alpha_k\tau}{h^2} \right) \times \\
\times \vartheta_3 \left(\frac{y-\eta}{2d}, \frac{i\pi\alpha_k\tau}{d^2} \right) f_l(\xi, \eta)\, d\xi\, d\eta,
\end{gathered}
$$

$$
\begin{gathered}
K_{kl} = \frac{1}{4hd} \int\limits_0^\tau d\theta \left\{ \vartheta_3 \left[\frac{h-x}{2h}, \frac{i\pi\alpha_k(\tau-\theta)}{h^2} \right] \times \right. \\
\left. \times \int\limits_{-d}^{+d} \vartheta_3 \left[\frac{y-\eta}{2d}, \frac{i\pi\alpha_s(\tau-\theta)}{d^2} \right] \psi_l(\eta, \alpha)\, d\eta \right\},
\end{gathered}
$$

$$
\begin{gathered}
M_{kl} = \frac{1}{4hd} \int\limits_0^\tau d\theta \left\{ \frac{\partial}{\partial y} \vartheta_3 \left[\frac{d-y}{2d}, \frac{i\pi\alpha_k(\tau-\theta)}{d^2} \right] \times \right. \\
\left. \times \int\limits_{-h}^{+h} \vartheta_3 \left[\frac{x-\xi}{2h}, \frac{i\pi\alpha_k(\tau-\theta)}{h^2} \right] \chi_l(\xi, \theta)\, d\xi \right\},
\end{gathered}
$$

$$
\alpha_k = d_q \mathrm{Lu}\, \nu_k^2.
$$

Problem 5. Solve the system of differential equations (1.1) and (1.2) with the following initial conditions:

$$
t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}
$$

and boundary conditions

$$
\frac{\partial t(0, y, \tau)}{\partial x} = t(x, 0, \tau) = 0, \tag{4}
$$

$$
\frac{\partial u(0, y, \tau)}{\partial x} = u(x, 0, \tau) = 0, \tag{5}
$$

$$-\lambda_q \frac{\partial t(h, y, \tau)}{dx} + \psi_1(y, \tau) = 0, \tag{6}$$

$$a_m \left[\frac{\partial u(h, y, \tau)}{\partial x} + \delta \frac{\partial t(h, y, \tau)}{\partial x}\right] + \psi_2(y, \tau) = 0, \tag{7}$$

$$-\lambda_q \frac{\partial t(x, d, \tau)}{\partial y} + \chi_1(x, \tau) = 0, \tag{8}$$

$$a_m \left[\frac{\partial u(x, d, \tau)}{\partial y} + \delta \frac{\partial t(x, d, \tau)}{\partial y}\right] + \chi_2(x, \tau) = 0. \tag{9}$$

Once again we utilize the two-dimensional, finite, complex Fourier transformation in the region $\Gamma(-h < x < +h, -d < y < +d)$

$$\bar{\varphi}(m, n) = \iint_{(\Gamma)} \exp\left[-im\frac{\pi}{h}\xi - i\left(n - \frac{1}{2}\right)\frac{\pi}{d}\eta\right] \times \\ \times \varphi(\xi, \eta)\, d\xi\, d\eta \tag{10}$$

and the corresponding inversion formula

$$\varphi(x, y) = \frac{1}{4hd} \sum_{(m, n)}^{-\infty, +\infty} \exp\left[im\frac{\pi}{h}x + i\left(n - \right.\right. \\ \left.\left. - \frac{1}{2}\right)\frac{\pi}{d}\eta\right]\bar{\varphi}(m, n). \tag{11}$$

Upon finding the Laplace transforms with respect to τ, we have a system of two algebraic equations of the first degree for two unknown functions $\bar{t}$ (m, n, s) and $\bar{u}$ (m, n, s). Solving this system and obtaining the inversion with respect to s, we get the same expressions as in Problem 1. Inverting with respect to m and n we get,

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + C_1 K_{11} + \\ + C_2 K_{21} + D_1 K_{12} + D_2 K_{22}, \tag{12}$$

$$I_{kl} = \frac{1}{4hd} \iint_{(\Gamma)} \vartheta_3\left(\frac{x - \xi}{2h}, \frac{i\pi a_k \tau}{h^2}\right) \vartheta_2 \times$$

$$\times\left(\frac{y-\eta}{2d}, \frac{i\pi\alpha_k\tau}{d^2}\right) f_l(\xi,\eta)\, d\xi\, d\eta, \tag{13}$$

$$K_{kl} = \frac{1}{4hd}\int_0 d\theta \left\{ \vartheta_3\left[\frac{h-x}{2h}, \frac{i\pi\alpha_k(\tau-\theta)}{h^2}\right] \times \right.$$

$$\times \int_{-d}^{+d} \vartheta_2\left[\frac{y-\eta}{2d}, \frac{i\pi\alpha_k(\tau-\alpha)}{\alpha_2}\right] \psi_l(\eta,\theta)\, d\eta\, +$$

$$+\,\vartheta_2\left[\frac{d-y}{2d}, \frac{i\pi\alpha_k(\tau-\theta)}{d^2}\right] \int_{-h}^{+h} \vartheta_3 \times$$

$$\left. \times \left[\frac{x-\xi}{2h}, \frac{i\pi\alpha_k(\tau-\theta)}{h^2}\right] \chi_l(\xi,\theta)\, d\xi \right\}. \tag{14}$$

The function $u(x,y,\tau)$ is given by equation (12) with $A_l \ldots$ replaced by $A_l' \ldots$ from equation (26) of Problem 1.

Problem 6. Solve the system of differential equations (1.1) and (1.2) with initial conditions

$$t(x,y,0) = f_1(x,y), \quad u(x,y,0) = f_2(x,y) \tag{3}$$

and boundary conditions

$$\frac{\partial t(0,y,\tau)}{\partial x} = t(x,0,\tau) = 0, \tag{4}$$

$$\frac{\partial u(0,y,\tau)}{\partial x} = u(x,0,\tau) = 0, \tag{5}$$

$$t(h,y,\tau) = \psi_1(y,\tau), \tag{6}$$

$$u(h,y,\tau) = \psi_2(y,\tau), \tag{7}$$

$$-\lambda_q \frac{\partial t(x,d,\tau)}{\partial y} + \chi_1(x,\tau) = 0, \tag{8}$$

$$a_m \frac{\partial u(x,d,\tau)}{\partial y} + a_m\delta \frac{\partial t(x,d,\tau)}{\partial y} + \chi_2(x,\tau) = 0. \tag{9}$$

In order to solve this problem, we make use of equations (10) and (11) of Problem 3. Continuing t and u into the interval, $(-h \leqslant x \leqslant 0)$

by the even method, and into the interval $(-d \leqslant y \leqslant 0)$ by the odd method, and transforming with respect to the spatial coordinates, we get the system of two ordinary differential equations (12) of Problem 2, where now,

$$\sigma = \pi^2 \left[\frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{\left(n - \frac{1}{2}\right)^2}{d^2} \right], \tag{10}$$

$$S_l(\tau) = (-1)^m i \int\limits_{-d}^{+d} \exp\left[-i\left(n - \frac{1}{2}\right)\frac{\pi}{d}\eta \right] \times$$

$$\times \left[i\left(m - \frac{1}{2}\right)\frac{\pi}{h} \right] \psi_l(\eta, \tau)\, d\eta, \tag{11}$$

$$I_l(\tau) = (-1)^n i \int\limits_{-h}^{+h} \exp\left[-i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\xi \right] \chi_l(\xi, \tau)\, d\xi. \tag{12}$$

The solution of this system is obtained from equation (15) of Problem 2.

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + E_1 K_{11} + E_2 K_{21} + a_m D_1 K_{12} +$$

$$+ a_m D_2 K_{22} + C_1 M_{11} + C_2 M_{21} + D_1 M_{12} + D_2 M_{22}, \tag{13}$$

where I_{kl}, K_{kl}, M_{kl} are given by equations (19), (20) and (21) of Problem 2. The function $u(x, y, \tau)$ is given by equation (13) with A_l replaced by A_l', as indicated in equation (17) of Problem 2.

Problem 7. Solve the system of differential equations (1.1) and (1.2) with the following initial conditions:

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and boundary conditions

$$\frac{\partial t(0, y, \tau)}{\partial x} = t(x, 0, \tau) = 0, \tag{4}$$

$$\frac{\partial u(0, y, \tau)}{\partial x} = u(x, 0, \tau) = 0, \tag{5}$$

$$t(h, y, \tau) = \psi_1(y, \tau), \tag{6}$$

$$u(h, y, \tau) = \psi_2(y, \tau), \tag{7}$$

$$t(x, d, \tau) = \chi_1(x, \tau), \tag{8}$$

$$u(x, d, \tau) = \chi_2(x, \tau). \tag{9}$$

To solve this problem we continue t and u into the interval ($-h \leqslant x \leqslant 0$) by the even method, and into the interval ($-d \leqslant y \leqslant 0$) by the odd method and we make use of the following formulas:

$$\bar{\varphi}(m, n) = \iint\limits_{(\Gamma)} \exp\left[-i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\xi - in\frac{\pi}{d}\eta\right]\varphi(\xi, \eta)d\xi\, dh, \tag{10}$$

$$\varphi(x, y) = \frac{1}{4hd} \sum_{(m,\,n)}^{-\infty,+\infty} \exp\left[i\left(m - \frac{1}{2}\right)\frac{\pi}{h}x + in\frac{\pi}{d}\eta\right]\bar{\varphi}(m, n). \tag{11}$$

Transforming with respect to the spatial coordinates, we get the system of two ordinary differential equations (12) of Problem 2, where now

$$\sigma = \pi^2 \cdot \left[\frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{h^2}{d^2}\right], \tag{12}$$

$$S_j(\tau) = 2(-1)^m i \int\limits_{-d}^{+d} \exp\left(-in\frac{\pi}{d}\eta\right)\left[i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\right] \times$$

$$\times \psi_j(\eta, \tau)\, d\eta + 2(-1)^n \int\limits_{-h}^{+h} \exp\left[-i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\xi\right] \times$$

$$\times\left(in\frac{\pi}{d}\right)\chi_j(\xi, \tau)\, d\xi. \tag{13}$$

The solution of this system is obtained by the usual method.

Inverting we get

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_2 I_{12} B_2 I_{22} + 2E_1 K_{11} +$$
$$+ 2E_2 K_{21} + a_m D_1 K_{12} + a_m D_2 K_{22}, \tag{14}$$

where I_{kl}, K_{kl} are determined by formulas (19) and (20) of Problem 3.

The coefficients A_i are the same as given in Problem 3. The function $u(x, y, \tau)$ is given by equation (14) with A_i, Bi... replaced by A'_i, B'_i ... of Problem 3.

Problem 8. Solve the system of differential equations (1.1) and (1.2) with the following initial conditions:

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y) \tag{3}$$

and boundary conditions

$$t(0, y, \tau) = t(x, 0, \tau) = 0, \tag{4}$$

$$u(0, y, \tau) = u(x, 0, \tau) = 0, \tag{5}$$

$$t(x, d, \tau) = \chi_1(x, \tau), \tag{6}$$

$$u(x, d, \tau) = \chi_2(x, \tau), \tag{7}$$

$$-\lambda_q \frac{\partial t(h, y, \tau)}{\partial x} + \psi_1(y, \tau) = 0, \tag{8}$$

$$a_m \frac{\partial u(h, y, \tau)}{\partial x} + a_m \delta \frac{\partial t(h, y, \tau)}{\partial x} + \psi_2(y, \tau) = 0. \tag{9}$$

To solve this problem we continue t and u into the intervals (-h$\leqslant$x$\leqslant$0) and (-d$\leqslant$y$\leqslant$0) by the odd method and use the formulas of Problem 2. Transforming with respect to the spatial coordinates, we get the system of two ordinary differential equations of Problem 2, where now

$$\sigma = \pi^2 \left[\frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{h^2}{d^2} \right], \tag{10}$$

$$S_j(\tau) = (-1)^n \int_{-h}^{+h} \exp \times$$

$$\times \left[-i\left(m - \frac{1}{2}\right) \frac{\pi}{h} \xi \right] \left(in \frac{\pi}{d} \right) \chi_1(\xi, \tau)\, d\xi, \tag{11}$$

$$I_j(\tau) = (-1)^m i \int_{-d}^{+d} \exp\left(-in \frac{\pi}{d} \eta \right) \psi_1(\eta, \tau)\, d\eta. \tag{12}$$

Solving this system and inverting, we get

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + E_1 K_{11} + E_2 K_{21} +$$

$$+a_mD_1K_{12}+a_mD_2K_{22}+C_1M_{11}+C_2M_{21}+D_1M_{12}+D_2M_{22}, \tag{13}$$

where

$$I_{kl}=\frac{1}{4hd}\iint\limits_{(\Gamma)}\vartheta_2\left(\frac{x-\xi}{2h},\ \frac{\alpha_k i\pi\tau}{h^2}\right)\times$$

$$\times\vartheta_3\left(\frac{y-\eta}{2d},\ \frac{\alpha_k i\pi\tau}{d^2}\right)f_l(\xi,\eta)\,d_\xi d_\eta, \tag{14}$$

$$K_{kl}=\frac{1}{4hd}\int\limits_0^\tau d\theta\left\{\frac{\partial}{\partial y}\vartheta_3\left[\frac{d-y}{2d}\ \frac{i\pi\alpha_k(\tau-\theta)}{d^2}\right]\times\right.$$

$$\left.\times\int\limits_{-h}^{+h}\vartheta_2\left[\frac{x-\xi}{2h},\ \frac{i\pi\alpha_k(\tau-\theta)}{h^2}\right]\chi_l(\xi,\theta)\,d_\xi\right\}, \tag{15}$$

$$M_{kl}=\frac{1}{4hd}\int\limits_0^\tau d\theta\left\{\vartheta_2\left[\frac{h-x}{2h},\ \frac{i\pi\alpha_k(\tau-\theta)}{h^2}\right]\times\right.$$

$$\left.\times\int\limits_{-d}^{+d}\vartheta_3\left[\frac{y-\eta}{2d},\ \frac{i\pi\alpha_k(\tau-\theta)}{d^2}\right]\psi_l(\eta,\theta)\,d_\eta\right\}, \tag{16}$$

$$\alpha_k=\nu_k^2\,a_q\mathrm{Lu}.$$

The coefficients A_l ... are given in Problem 2. The function u(x, y, τ) is given by equation (13) with A_l, B_l... replaced by A_l', B_l of Problem 2.

Problem 9. Solve the system of differential equations (1.1) and (1.2) with initial conditions

$$t(x,y,0)=f_1(x,y),\ u(x,y,0)=f_2(x,y) \tag{3}$$

and boundary conditions

$$t(0,y,\tau)=t(x,0,\tau)=0, \tag{4}$$

$$u(0,y,\tau)=u(x,0,\tau)=0, \tag{5}$$

$$-\lambda_q \frac{\partial t(h, y, \tau)}{\partial x} + \psi_1(y, \tau) = 0, \tag{6}$$

$$a_m \frac{\partial u(h, y, \tau)}{\partial x} + a_m\delta \frac{\partial t(h, y, \tau)}{\partial x} + \psi_2(y, \tau) = 0, \tag{7}$$

$$-\lambda_q \frac{\partial t(x, d, \tau)}{\partial y} + \chi_1(x, \tau) = 0, \tag{8}$$

$$a_m \frac{\partial u(x, d, \tau)}{\partial y} + a_m\delta \frac{\partial t(x, d, \tau)}{\partial y} + \chi_2(x, \tau) = 0. \tag{9}$$

The solution of this problem with boundary conditions of the first and second kind is obtained by utilizing the following formulas:

$$\bar{\varphi}(m, n) = \iint\limits_{(\Gamma)} \exp\left[-i\left(m - \frac{1}{2}\right)\frac{\pi}{h}\xi - \right.$$

$$\left. - i\left(n - \frac{1}{2}\right)\frac{\pi}{d}\eta\right] \varphi(\xi, \eta)\, d\xi,\, d\eta, \tag{10}$$

$$\varphi(x, y) = \frac{1}{4hd} \sum_{m, n}^{-\infty, +\infty} \exp\left[i\left(m - \frac{1}{2}\right)\frac{\pi}{h}x + \right.$$

$$\left. + i\left(n - \frac{1}{2}\right)\frac{\pi}{d}y\right] \bar{\varphi}(m, n). \tag{11}$$

Transforming with respect to the spatial coordinates, we get the system of two ordinary differential equations of Problem 1, where now

$$\sigma = \left[\frac{\left(m - \frac{1}{2}\right)^2}{h^2} + \frac{\left(n - \frac{1}{2}\right)^2}{d^2}\right], \tag{12}$$

$$S_j(\tau) = (-1)^m i \int\limits_{-d}^{+d} \exp\left[-i\left(n - \frac{1}{2}\right)\frac{\pi}{d}\eta\right] \psi_1(\eta, \tau)\, d\eta +$$

$$+(-1)^n i \int_{-h}^{+h} \exp\left[-i\left(m-\frac{1}{2}\right)\frac{\pi}{h}\xi\right]\chi_1(\xi, \tau)\, d\xi. \tag{13}$$

Solving this system and inverting, we get

$$t(x, y, \tau) = A_1 I_{11} + A_2 I_{21} + B_1 I_{12} + B_2 I_{22} + C_1 K_{11} + \\ + C_2 K_{12} + D_1 K_{12} + D_2 K_{22}, \tag{14}$$

where

$$I_{kl} = \frac{1}{4hd} \iint_{(\Gamma)} \vartheta_2\left(\frac{x-\xi}{2h}, \frac{\pi\alpha_k\tau}{h^2}\right) \times \\ \times \vartheta_2\left(\frac{y-\eta}{2d}, \frac{i\pi\alpha_k\tau}{d^2}\right) f_l(\xi, \eta)\, d\xi\, d\eta, \tag{15}$$

$$K_{kl} = \frac{1}{4hd} \int_0^\tau d\theta \left\{\vartheta_2\left[\frac{h-x}{2h}, \frac{i\pi\alpha_k(\tau-\theta)}{h^2}\right] \times \right. \\ \times \int_{-d}^{+d} \vartheta_2\left[\frac{y-\eta}{2d}, \frac{i\pi\alpha_k(\tau-\theta)}{d^2}\right]\psi_l(\eta, \theta)\, d\eta + \\ + \vartheta_2\left[\frac{d-y}{2d}, \frac{i\pi\alpha_k(\tau-\theta)}{d^2}\right] \times \\ \left. \times \int_{-h}^{+h} \vartheta_2\left[\frac{x-\xi}{2h}, \frac{i\pi\alpha_k(\tau-\theta)}{h^2}\right]\chi_l(\xi, \theta)\, d\xi \right\}. \tag{16}$$

The coefficients $A_l \ldots$ are given in Problem 1. The function $u(x, y, \tau)$ is given by equation (14), with A_l replaced by $A_l'\, B_l'$ of Problem 1.

Problem 10. Solve the system of differential equations (1.1) and (1.2) with initial conditions

$$t(x, y, 0) = f_1(x, y), \quad u(x, y, 0) = f_2(x, y)$$

and boundary conditions

$$t(0, y, \tau) = t(x, 0, \tau) = 0,$$

$$u(0,\ y,\ \tau) = u(x,\ 0,\ \tau) = 0, \tag{5}$$

$$t(h,\ y,\ \tau) = \psi_1(y,\ \tau), \tag{6}$$

$$u(h,\ y,\ \tau) = \psi_2(y,\ \tau), \tag{7}$$

$$t(x,\ d,\ \tau) = \chi_1(x,\ \tau), \tag{8}$$

$$u(x,\ d,\ \tau) = \chi_2(x,\ \tau). \tag{9}$$

To solve this problem with boundary conditions of the first kind we employ the formulas used in Problem 1. Transforming with respect to the spatial coordinates, we get the system of two ordinary differential equations of Problem 3, where now

$$\sigma = \pi^2 \left(\frac{m^2}{h^2} + \frac{n^2}{d^2} \right), \tag{10}$$

$$\begin{gathered} S_l(\tau) = 2(-1)^m \int\limits_{-d}^{+d} \exp\left(-in\frac{\pi}{d}\eta \right) \left(im\frac{\pi}{h} \right) \times \\ \times \psi_l(\eta,\ \tau)\,d\eta + 2(-1)^n \int\limits_{-h}^{+h} \exp\left(-im\frac{\pi}{h}\xi \right) \times \\ \times \left(in\frac{\pi}{d} \right) \chi_l(\xi,\ \tau)\,d\xi. \end{gathered} \tag{11}$$

Solving this system and inverting, we get

$$\begin{gathered} t(x,\ y,\ \tau) = A_{11}I_{11} + A_2I_{21} + B_1I_{12} + B_2I_{22} + 2E_1K_{11} + \\ + 2E_2K_{21} + a_mD_1K_{12} + a_mD_2K_{22}, \end{gathered} \tag{12}$$

where

$$\begin{gathered} I_{kl} = \frac{1}{4hd} \iint\limits_{(\Gamma)} \vartheta_3 \left(\frac{x-\xi}{2h},\ \frac{i\pi\alpha_k\tau}{h^2} \right) \times \\ \times \vartheta_3 \left(\frac{y-\eta}{2d},\ \frac{i\pi\alpha_k\tau}{d^2} \right) f_l\,(\xi,\ \eta)\,d\xi\,d\eta, \end{gathered} \tag{13}$$

$$K_{kl} = \frac{1}{2hd} \left[\int\limits_0^{\tau} d\theta \left\{ \frac{\partial}{\partial x} \vartheta_3 \left[\frac{h-x}{2h},\ \frac{i\pi\alpha_k(\tau-\theta)}{h^2} \right] \times \right. \right.$$

$$\times \int_{-d}^{+d} \vartheta_3 \left[\frac{y-\eta}{2d}, \frac{i\pi\alpha_k(\tau-\theta)}{d^2} \right] \psi_l(\eta, \theta)\, d\eta +$$

$$+ \frac{\partial}{\partial y} \vartheta_3 \left[\frac{d-y}{2d}, \frac{i\pi\alpha_k(\tau-\theta)}{d^2} \right] \times$$

$$\times \int_{-h}^{+h} \vartheta_3 \left[\frac{x-\xi}{2h}, \frac{i\pi\alpha_k(\tau-\theta)}{h^2} \right] \chi_l(\xi, \theta)\, d\xi \Big\} \Big] . \qquad (14)$$

The coefficients $A_l B_l$ are given in Problem 3. The function u(x, y, τ) is given by equation (12) with $A_l' B_l'$ replaced by A_l, B_l of Problem 3.

CHAPTER VI

Heat and Mass Transfer with Both Molecular and Molar Transport of Energy and Mass

The experimental investigations of P. D. Lebedev, G. A. Maksimov, V. V. Krasnikov and others show that during intensive heating processes a molar (filtration) transfer of heat and mass can occur in a porous medium under the influence of hydrodynamic forces. This molar, or macroscopic, transfer occurs in addition to the molecular heat and mass transfer. To such phenomena belong the motion of liquid caused by expansion of confined air and the filtration-like movement of steam caused by the non-relaxing pressure gradient of a steam-gas mixture. The superposition of macroscopic transport upon molecular processes taking place in a medium, affects the transfer mechanism and causes a substantial intensification of the process. For example, in drying cellulose by high frequency current and infra-red rays, about 50% of the moisture is transferred to the surface in the form of vapor as a result of the internal pressure excess. As a first approximation, molar mass transfer in a porous medium can be described by d'Arcy's law, according to which the molar, or macroscopic, flux of mass is proportional to the pressure gradient and inversely proportional to the kinematic viscosity; the proportionality constant is called the coefficient of permeability. Thus we can regard the quotient of the latter by the kinematic viscosity as a transfer coefficient for macroscopic mass transfer, and we can regard the pressure as the molar or filtration transfer potential. Then the system of differential equations for heat and mass transfer with both molecular and macroscopic transport of energy and mass can be represented in the following form:

$$
\begin{aligned}
c_q \gamma \frac{\partial t}{d\tau} &= \nabla(\lambda_q \nabla t) + \varepsilon \rho c'_m \gamma \frac{\partial \theta}{\partial \tau} - \sum_i c_i q_{mi} \nabla t, \\
c'_m \gamma \frac{\partial \theta}{\partial \tau} &= \nabla [\lambda_m (\nabla \theta + \delta \nabla t) + \lambda_p \nabla p], \qquad \text{(XVIII)} \\
c_p \gamma \frac{\partial p}{\partial \tau} &= \nabla(\lambda_p \nabla p) - \varepsilon c'_m \gamma \frac{\partial \theta}{\partial \tau}.
\end{aligned}
$$

Here, λ_q is the thermal conductivity, λ_p is the coefficient of macroscopic mass transfer, c_p is the specific "vapor capacity" of the medium, p is the "filtration" potential, and c'_m is the specific "mass capacity." The latter is the derivative of the specific content (concentration) of bound matter with respect to the mass transfer potential; in a moist, porous body the isothermal specific mass capacity is the partial derivative at constant temperature of the ratio of moisture to skeletal mass with respect to the mass-transfer potential. The remaining notation has been presented in previous chapters.

For simplicity, the system of equations will be arranged in dimensionless form. The following version is applicable to computation, by zones, of high intensity, one-dimensional heat and mass transfer:

$$\frac{\partial T(X, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 T(X, \mathrm{Fo})}{\partial X^2} - \varepsilon \mathrm{Ko} \frac{\partial \Theta(X, \mathrm{Fo})}{\partial \mathrm{Fo}}, \qquad \text{(XVIII')}$$

$$\frac{\partial \Theta(X, \mathrm{Fo})}{\partial \mathrm{Fo}} = \mathrm{Lu} \frac{\partial^2 \Theta(X, \mathrm{Fo})}{\partial X^2} -$$

$$- \mathrm{LuPn} \frac{\partial^2 T(X, \mathrm{Fo})}{\partial X^2} - \mathrm{Lu}_p \frac{\mathrm{Bu}}{\mathrm{Ko}} \frac{\partial^2 P(X, \mathrm{Fo})}{\partial X^2},$$

$$\frac{\partial P(X, \mathrm{Fo})}{\partial \mathrm{Fo}} = \mathrm{Lu}_p \frac{\partial^2 P(X, \mathrm{Fo})}{\partial X^2} + \varepsilon \frac{\mathrm{Ko}}{\mathrm{Bu}} \frac{\partial \Theta(X, \mathrm{Fo})}{\partial \mathrm{Fo}},$$

where

$\mathrm{Bu} = \frac{\rho c_p}{c_q} \frac{p_0}{t_c - t_0}$ — is the Bulygin number, characterizing the "accumulation" capacity of the body;

$\mathrm{Lu}_p = \frac{a_p}{a_q}$ — is the "inertia" number of the pressure field with respect to the temperature field (a_p is the conductivity coefficient of filtration potential);

$X = \frac{x}{R}$ — is the dimensionless coordinate;

T, Θ, P — are dimensionless potentials of heat, mass and filtration transfer respectively; for uniform initial distribution of the potentials:

$$T = \frac{t - t_0}{t_c - t_0}, \quad \Theta = \frac{\theta_0 - \theta}{\theta_0 - \theta_p}, \quad P = \frac{p - p_0}{p_0}.$$

1. NON-STATIONARY FIELDS OF ENERGY AND MASS-TRANSFER POTENTIALS WITH BOUNDARY CONDITIONS OF THE THIRD KIND

We shall now find the unsteady-state distributions of dimensionless potentials in porous media whose shapes approximate either an infinite plate or a sphere. We assume that the problem is symmetrical, and the initial distributions of the potentials throughout the material are uniform. Then the boundary conditions are

$$\frac{\partial T(1,\ \mathrm{Fo})}{\partial X} - \mathrm{Bi}_q\,[1-T(1,\ \mathrm{Fo})] + (1-\varepsilon)\,\mathrm{LuKoBi}_m\,[1-\Theta(1,\ \mathrm{Fo})] = 0, \tag{1.1}$$

$$-\frac{\partial \Theta(1,\ \mathrm{Fo})}{\partial X} + \mathrm{Pn}\frac{\partial T(1,\ \mathrm{Fo})}{\partial X} + \frac{\mathrm{Bu}}{\mathrm{Ko}}\,\frac{\mathrm{Lu}_{p1}}{\mathrm{Lu}}\,\frac{\partial P(1,\ \mathrm{Fo})}{\partial X} + \mathrm{Bi}_m\,[1-\Theta(1,\ \mathrm{Fo})] = 0, \tag{1.2}$$

$$P(1,\ \mathrm{Fo}) = 0, \tag{1.3}$$

$$\frac{\partial T(0,\ \mathrm{Fo})}{\partial X} = \frac{\partial \Theta(0,\ \mathrm{Fo})}{\partial X} = \frac{\partial P(0,\ \mathrm{Fo})}{\partial X} = 0, \tag{1.4}$$

$$T(X,\ 0) = 0, \quad \Theta(X,\ 0) = 0, \quad P(X,\ 0) = 0. \tag{1.5}$$

a) Infinite Plate.

Let us denote

$$F_L(X,\ s) = \int_0^\infty F(X,\ \mathrm{Fo})\exp(-s\,\mathrm{Fo})\,d\mathrm{Fo}. \tag{1.6}$$

Applying the integral Laplace transformation to equations (XVIII') and considering initial conditions (1.5), we get

$$sT_L = T''_L - \varepsilon\,\mathrm{Ko}\,s\,\Theta_L, \tag{1.7}$$

$$s\,\Theta_L = \mathrm{Lu}\,\Theta''_L - \mathrm{LuPn}T''_L - \mathrm{Lu}_p\frac{\mathrm{Bu}}{\mathrm{Ko}}P''_L, \tag{1.8}$$

$$sP_L = \mathrm{Lu}_p P''_L + \varepsilon\frac{\mathrm{Ko}}{\mathrm{Bu}}\,s\,\Theta_L. \tag{1.9}$$

The solutions of the system of ordinary differential equations (1.7) to (1.9) in terms of transforms can be obtained by the following method. From equation (1.7) we determine Θ_L and substitute it into equations (1.8) and (1.9). As a result, we get a system of two equations in T_L and P_L. The solution of the new system is found by means of a second Laplace transformation with respect to the dimensionless coordinate. The solution for Θ_L is then found by substitution of the expressions for T_L and P_L, and their second derivatives into equation (1.7).

$$\begin{aligned} T_L = {} & A_1 \exp(\sqrt{s}\,\nu_1 X) + A_2 \exp(\sqrt{s}\,\nu_2 X) + \\ & + A_3 \exp(\sqrt{s}\,\nu_3 X) + A_4 \exp(-\sqrt{s}\,\nu_1 X) + \\ & + A_5 \exp(-\sqrt{s}\,\nu_2 X) + A_6 \exp(-\sqrt{s}\,\nu_3 X). \end{aligned} \tag{1.10}$$

$$\begin{aligned} \Theta_L = {} & -\frac{1}{\varepsilon \mathrm{Ko}} [A_1 (1 - \nu_1^2) \exp(\sqrt{s}\,\nu_1 X) + \\ & + A_2 (1 - \nu_2^2) \exp(\sqrt{s}\,\nu_2 X) + A_3 (1 - \nu_3^2) \exp(\sqrt{s}\,\nu_3 X) + \\ & + A_4 (1 - \nu_1^2) \exp(-\sqrt{s}\,\nu_1 X) + A_5 (1 - \nu_2^2) \exp(-\sqrt{s}\,\nu_2 X) + \\ & + A_6 (1 - \nu_3^2) \exp(-\sqrt{s}\,\nu_3 X)], \end{aligned} \tag{1.11}$$

$$\begin{aligned} P_L = {} & \frac{1}{\varepsilon \mathrm{Bu}} [A_1 \sigma_1 \exp(\sqrt{s}\,\nu_1 X) + A_2 \sigma_2 \exp(\sqrt{s}\,\nu_2 X) + \\ & + A_3 \sigma_3 \exp(\sqrt{s}\,\nu_3 X) + A_4 \sigma_1 \exp(-\sqrt{s}\,\nu_1 X) + \\ & + A_5 \sigma_2 \exp(-\sqrt{s}\,\nu_2 X) + A_6 \sigma_3 \exp(-\sqrt{s}\,\nu_3 X)]. \end{aligned} \tag{1.12}$$

Here,

$$\sigma_j = (1 - \varepsilon)(1 - \nu_j^2) - \mathrm{Lu}(1 - \nu_j^2)\nu_j^2 - \varepsilon\, \mathrm{KoPnLu}\, \nu_j^2, \tag{1.13}$$

$$\nu_j = \sqrt{\Lambda_j + \frac{1}{3}\alpha}, \quad (j = 1, 2, 3). \tag{1.14}$$

The Λ_j appearing in equation (1.14) are determined by

$$\begin{aligned} \Lambda = {} & \sqrt[3]{-\frac{1}{2}\Pi_2 + \sqrt{\frac{1}{4}\Pi_2^2 + \frac{1}{27}\Pi_1^3}} + \\ & + \sqrt[3]{-\frac{1}{2}\Pi_2 - \sqrt{\frac{1}{4}\Pi_2^2 + \frac{1}{27}\Pi_1^3}}. \end{aligned}$$

The correct values of Λ should at the same time satisfy the following relationship:

$$\sqrt[3]{-\frac{1}{2}\Pi_2+\sqrt{\frac{1}{4}\Pi_2^2+\frac{1}{27}\Pi_1^3}}\times$$

$$\times\sqrt[3]{-\frac{1}{2}\Pi_2-\sqrt{\frac{1}{4}\Pi_2^2+\frac{1}{27}\Pi_1^3}}=-\frac{1}{3}\Pi_1,$$

where,

$$\Pi_1=-\frac{\alpha^2}{3}+\beta,\quad \Pi_2=-\frac{2\,\alpha^8}{27}+\frac{\alpha\beta}{3}-\gamma,$$

$$\alpha=1+(1-\varepsilon)\frac{1}{\mathrm{Lu}}+\frac{1}{\mathrm{Lu}_p}+\varepsilon\,\mathrm{KoPn},$$

$$\beta=(1-\varepsilon)\frac{1}{\mathrm{Lu}}+\left(1+\frac{1}{\mathrm{Lu}}+\varepsilon\,\mathrm{KoPn}\right)\frac{1}{\mathrm{Lu}_p},$$

$$\gamma=\frac{1}{\mathrm{Lu}\mathrm{Lu}_p}.$$

The values of ν_j for various values of the dimensionless numbers of heat and mass transfer are given in Table 36.

Thus, the general solution of the system of equations (XVIII') in terms of transforms can be described by equations (1.10) to (1.12), where $A_k\,(k=1, 2, \ldots, 6)$ are constant coefficients with respect to X. The values of the latter are found from boundary conditions (1.1) to (1.4). First, utilize the conditions of symmetry (1.4). Following with the Laplace transformation (1.6) they become:

$$T'_L(0, s)=\Theta'_L(0, s)=P'_L(0, s)=0.$$

In order that the general solutions satisfy the conditions of symmetry we must have

$$A_1=A_4,\quad A_2=A_5,\quad A_3=A_6.$$

On the basis of these relationships, the solutions (1.10) to (1.12) can be rewritten in the following form:

$$T_L(X, s)=\sum_{j=1}^{3}C_j\,\mathrm{ch}\sqrt{s}\,\nu_j X, \tag{1.15}$$

Table 36

$\nu_j = \nu_j(\mathrm{Lu}_p, \mathrm{Lu}, \varepsilon, \mathrm{KoPn})$ $j = 1, 2, 3$

ν_j	Lu_p					Lu			
	100	300	500	700	1000	0,3	0,7	1,0	3,333
$j = 1$	1,8070	1,8080	1,8080	1,8080	1,8080	1,8080	1,6480	1,6100	1,5510
$j = 2$	0,1892	0,0938	0.0825	0,0686	0,0583	0,0825	0,2830	0,3135	0,9801
$j = 3$	0,5345	0,5502	0,5498	0,5507	0,5514	0,5498	0,4570	0,4312	0,3734
Values of other parameters	$\mathrm{Lu} = 0{,}3$, $\varepsilon = 0{,}7$, $\mathrm{KoPn} = 2{,}25$					$\mathrm{Lu}_p = 500$, $\varepsilon = 0{,}7$, $\mathrm{KoPn} = 2{,}25$			

ν_j	ε				KoPn						
	0,1	0,4	0,7	0,9	0,45	0,75	1,50	2,25	3,0	4,05	5,85
$j = 1$	1,8230	1,8150	1,8080	1.8040	1,3190	1,4250	1,6360	1,8080	1,9590	2,1490	2,4340
$j = 2$	0,0469	0,0577	0,0825	0,1604	0,0820	0,0820	0,0820	0,0825	0,0820	0,0820	0,0849
$j = 3$	0,9504	0,7790	0,5498	0,2828	0,7560	0,6990	0,6084	0,5498	0,5066	0,4611	0,4053
Values of other parameters	$\mathrm{Lu} = 0{,}3$, $\mathrm{Lu}_p = 500$, $\mathrm{KoPn} = 2{,}25$				$\mathrm{Lu} = 0{,}3$, $\mathrm{Lu}_p = 500$, $\varepsilon = 0{,}7$						

$$\Theta_L(X, s) = -\frac{1}{\varepsilon\,\mathrm{Ko}} \sum_{j=1}^{3} C_j (1 - \nu_j^2)\, \mathrm{ch}\sqrt{s}\,\nu_j X, \tag{1.16}$$

$$P_2(X, s) = \frac{1}{\varepsilon\,\mathrm{Bu}} \sum_{j=1}^{3} C_j \sigma_j\, \mathrm{ch}\sqrt{s}\,\nu_j X, \tag{1.17}$$

where C_j $(j = 1, 2, 3)$ are new constants with respect to the coordinates. Applying the transformation (1.6) to the boundary conditions (1.1) and (1.2), we get

$$\begin{gathered} T_L'(1, s) - \mathrm{Bi}_q \left[\frac{1}{s} - T_L(1, s)\right] + \\ + (1 - \varepsilon)\,\mathrm{Lu}\,\mathrm{Ko}\,\mathrm{Bi}_m \left[\frac{1}{s} - \Theta_L(1, s)\right] = 0, \end{gathered} \tag{1.18}$$

$$\begin{gathered} -\Theta_L'(1, s) + \mathrm{Pn}\, T_E'(1, s) + \\ + \frac{\mathrm{Bu}}{\mathrm{Ko}} \frac{\mathrm{Lu}_p}{\mathrm{Lu}} P_L'(1, s) + \mathrm{Bi}_m \left[\frac{1}{s} - \Theta_L(1, s)\right] = 0, \end{gathered} \tag{1.19}$$

$$P_L(1, s) = 0. \tag{1.20}$$

Substituting the solutions (1.15) to (1.17) into (1.18) to (1.20) we get the system of algebraic equations

$$\left.\begin{aligned} &\sum_{j=1}^{3} C_j P_j + \frac{\varepsilon\,\mathrm{Ko}}{s} = 0 \\ &\sum_{j=1}^{3} C_j Q_j + \frac{\varepsilon\,\mathrm{Ko}\,K_1 - 1}{s} = 0 \\ &\sum_{j=1}^{3} C_j \chi_j = 0 \end{aligned}\right\}, \tag{1.21}$$

solving the above yields

$$C_1 = \frac{(\varepsilon\,\mathrm{Ko}\,K_1 - 1)\left(P_2 - \frac{\chi_2}{\chi_3} P_3\right) - \varepsilon\,\mathrm{Ko}\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right)}{s\left[\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right)\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right) - \left(P_2 - \frac{\chi_2}{\chi_3} P_3\right)\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right)\right]}, \tag{1.22}$$

$$C_2 = -\frac{(\varepsilon \mathrm{Ko}\, K_1 - 1)\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right) - \varepsilon \mathrm{Ko}\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right)}{s\left[\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right)\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right) - \left(P_2 - \frac{\chi_2}{\chi_3} P_3\right)\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right)\right]}, \quad (1.23)$$

$$C_3 = \frac{(\varepsilon \mathrm{Ko}\, K_1 - 1)\left(P_1 \frac{\chi_2}{\chi_3} - P_2 \frac{\chi_1}{\chi_3}\right) - \varepsilon \mathrm{Ko}\left(Q_1 \frac{\chi_2}{\chi_3} - Q_2 \frac{\chi_1}{\chi_3}\right)}{s\left[\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right)\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right) - \left(P_2 - \frac{\chi_2}{\chi_3} P_3\right)\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right)\right]}, \quad (1.24)$$

where

$$P_j = (1 - \nu_j^2)\, \mathrm{ch} \sqrt{s}\, \nu_j +$$

$$+ \frac{(1 - \nu_j^2) + \varepsilon \mathrm{Ko}\, \mathrm{Pn} + \frac{\mathrm{Lu}_p}{\mathrm{Lu}} \sigma_j}{\mathrm{Bi}_m} \sqrt{s}\, \nu_j\, \mathrm{sh} \sqrt{s}\, \nu_j,$$

$$Q_j = [1 + (1 - \nu_j^2) K_1]\, \mathrm{ch} \sqrt{s}\, \nu_j + \frac{1}{\mathrm{Bi}_q} \sqrt{s}\, \nu_j\, \mathrm{sh} \sqrt{s}\, \nu_j,$$

$$\chi_j = \sigma_j\, \mathrm{ch} \sqrt{s}\, \nu_j.$$

In order to obtain the inversion, each of the solutions should be represented as a ratio of two generalized polynomials in s: $\frac{\Phi(s)}{\psi(s)}$. The common denominator of the respective ratios for T_L, Θ_L and P_L is

$$\psi(s) = s\left[\left(P_1 - \frac{\chi_1}{\chi_3} P_3\right)\left(Q_2 - \frac{\chi_2}{\chi_3} Q_3\right) - \right.$$

$$\left. - \left(P_2 - \frac{\chi_2}{\chi_3} P_3\right)\left(Q_1 - \frac{\chi_1}{\chi_3} Q_3\right)\right]. \quad (1.25)$$

Analysis of the ratios and of the roots of the denominator of equation (1.25) shows that the necessary conditions for inversion by the expansion theorem are fulfilled and that we can utilize the following formula:

$$L^{-1}\left[\frac{\Phi(s)}{\psi(s)}\right] = \frac{\Phi(0)}{\psi'(0)} + \sum_{n=1}^{\infty} \frac{\Phi(s_n)}{\psi'(s_n)} \exp(s_n \mathrm{Fo}), \quad (1.26)$$

where, $s = 0$ and $s = s_n$ are the roots of equation (1.25).

Determining $\psi'(0)$, $\Phi(0)$, $\psi'(s_n)$, $\Phi(s_n)$ for the distribution of dimensionless transfer potentials and substituting the values obtained into equation (1.26), we get the final solution of the problem:

$$T(X, \mathrm{Fo}) = 1 - \sum_{n=1}^{\infty} \sum_{j=1}^{3} C_{nj} \cos \nu_j \mu_n X \exp\left(-\mu_n^2 \mathrm{Fo}\right), \tag{1.27}$$

$$\Theta(X, \mathrm{Fo}) = 1 + \frac{1}{\varepsilon \mathrm{Ko}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} \times$$

$$\times C_{nj}\left(1 - \nu_j^2\right) \cos \nu_j \mu_n X \exp\left(-\mu_n^2 \mathrm{Fo}\right), \tag{1.28}$$

$$P(X, \mathrm{Fo}) = -\frac{1}{\varepsilon \mathrm{Bu}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} \times$$

$$\times C_{nj} \sigma_j \cos \nu_j \mu_n X \exp\left(-\mu_n^2 \mathrm{Fo}\right). \tag{1.29}$$

The coefficients C_{nj} are determined by

$$C_{n1} = 2\left(\varepsilon \mathrm{Ko} K_1 - 1\right)\left(P_{n2} - \frac{\chi_{n2}}{\chi_{n3}} P_{n3}\right) -$$

$$- \varepsilon \mathrm{Ko}\left(Q_{n2} - \frac{\chi_{n2}}{\chi_{n3}} Q_{n3}\right)\left(\mu_n \psi_n\right)^{-1}, \tag{1.30}$$

$$C_{n2} = -2\left(\varepsilon \mathrm{Ko} K_1 - 1\right)\left(P_{n1} - \frac{\chi_{n1}}{\chi_{n3}} P_{n3}\right) -$$

$$- \varepsilon \mathrm{Ko}\left(Q_{n1} - \frac{\chi_{n1}}{\chi_{n2}} Q_{n3}\right)\left(\mu_n \psi_n\right)^{-1}, \tag{1.31}$$

$$C_{n3} = 2\left(\varepsilon \mathrm{Ko} K_1 - 1\right)\left(P_{n1} \frac{\chi_{n2}}{\chi_{n3}} - P_{n2} \frac{\chi_{n1}}{\chi_{n3}}\right) \cdot -$$

$$- \varepsilon \mathrm{Ko}\left(Q_{n1} \frac{\chi_{n2}}{\chi_{n3}} - Q_{n2} \frac{\chi_{n1}}{\chi_{n2}}\right)\left(\mu_n \psi_n\right)^{-1}, \tag{1.32}$$

where,

$$\begin{aligned}\psi_n = {} & \left(P_{n1}-\frac{\chi_{n1}}{\chi_{n3}}P_{n3}\right)\left[\nu_2A_{n2}-\frac{\chi_{n2}}{\chi_{n3}}\left(\nu_3A_{n3}+b_{n2}Q_{n3}\right)\right]+\\ & +\left[\nu_1B_{n1}-\frac{\chi_{n1}}{\chi_{n3}}\left(\nu_3B_{n3}+b_{n1}P_{n3}\right)\right]\left(Q_{n2}-\frac{\chi_{n2}}{\chi_{n3}}Q_{n3}\right)-\\ & -\left(P_{n2}-\frac{\chi_{n2}}{\chi_{n3}}P_{n3}\right)\left[\nu_1A_{n1}-\frac{\chi_{n1}}{\chi_{n3}}\left(\nu_3A_{n3}+b_{n1}Q_{n3}\right)\right]-\\ & -\left[\nu_2B_{n2}-\frac{\chi_{n2}}{\chi_{n3}}\left(\nu_3B_{n3}+b_{n2}P_{n3}\right)\right]\left(Q_{n1}-\frac{\chi_{n1}}{\chi_{n3}}Q_{n3}\right),\end{aligned} \qquad (1.33)$$

$$A_{nj}=\left[1+\frac{1}{\mathrm{Bi}_q}+\left(1-\nu_j^2\right)K_1\right]\sin\nu_j\mu_n+\frac{\nu_j\mu_n}{\mathrm{Bi}_q}\cos\nu_j\mu_n, \qquad (1.34)$$

$$\begin{aligned}B_{nj}= {} & \left(1-\nu_j^2\right)\sin\nu_j\mu_n+\frac{\left(1-\nu_j^2\right)+\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}+\dfrac{\mathrm{Lu}_p}{\mathrm{Lu}}\sigma_j}{\mathrm{Bi}_m}\times\\ & \times\left(\sin\nu_j\mu_n+\nu_j\mu_n\cos\nu_j\mu_n\right),\end{aligned} \qquad (1.35)$$

$$b_{nj}=\nu_j\,\mathrm{tg}\,\nu_j\mu_n-\nu_3\,\mathrm{tg}\,\nu_3\mu_n, \qquad (1.36)$$

$$\begin{aligned}P_{nj}= {} & \left(1-\nu_j^2\right)\cos\nu_j\mu_n-\frac{\left(1-\nu_j^2\right)+\varepsilon\,\mathrm{Ko}\,\mathrm{Pn}+\dfrac{\mathrm{Lu}_p}{\mathrm{Lu}}\sigma_j}{\mathrm{Bi}_m}\times\\ & \times\nu_j\mu_n\sin\nu_j\mu_n,\end{aligned} \qquad (1.37)$$

$$Q_{nj}=\left[1+\left(1-\nu_j^2\right)K_1\right]\cos\nu_j\mu_n-\frac{\nu_j\mu_n}{\mathrm{Bi}_q}\sin\nu_j\mu_n, \qquad (1.38)$$

$$\chi_{nj}=\sigma_j\cos\nu_j\mu_n. \qquad (1.39)$$

The characteristic roots $\mu_n=i\sqrt{s_n}$ are determined by the solutions of the following equations:

$$\left(P_1-\frac{\chi_1}{\chi_3}P_3\right)\left(Q_2-\frac{\chi_2}{\chi_3}Q_3\right)-\left(P_2-\frac{\chi_2}{\chi_3}P_3\right)\left(Q_1-\frac{\chi_1}{\chi_3}Q_3\right)=0. \qquad (1.40)$$

Tables 37 to 42 and Figures 48 to 52 give the values of the characteristic roots, μ_n for various dimensionless numbers as a function of Bi_q. The values of μ_n were obtained by graphical solution of equation (1.40).

Table 37. Roots of characteristic equation (1.40) as a function of Bi_m ($Lu = 0{,}3$, $\varepsilon = 0{,}7$, $KoPn = 2{,}25$, $Lu_p = 500$)

Bi_q	$Bi_m = 5$			$Bi_m = 10$			$Bi_m = 15$			$Bi_m = 20$		
	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
0 30	0,454	1,221	1,702	0,466	—	—	0,468	—	—	0,469	—	—
0,35	0,476	1,358	1,700	0.492	—	—	0,494	—	—	0,495	—	—
0,40	0,493	1,392	1,697	0,513	—	—	0,515	—	—	0,517	—	—
0,45	0,511	1,429	1,686	0.532	—	—	0,534	—	—	0,536	—	—
0,50	0,528	1,462	1,668	0,552	—	—	0,553	—	—	0,556	—	—
0,60	0,555	—	—	0,577	—	—	0,581	—	—	0,583	—	—
0,70	0,578	—	—	0.599	—	—	0,605	—	—	0,607	—	—
0,80	0,596	—	—	0,617	—	—	0,624	—	—	0,627	—	—
0,90	0,611	—	—	0,633	—	—	0,640	—	—	0,645	—	—
1,00	0,622	—	—	0 645	—	—	0,653	—	—	0,658	—	—
1,50	0,656	—	—	0,686	—	—	0,700	—	—	0,706	—	—
2,00	0,676	—	—	0,714	—	—	0,725	—	—	0,733	—	—
3,00	0,696	—	—	0,739	—	—	0,755	—	—	0,764	—	—
4,00	0,704	—	—	0,752	—	—	0,771	—	—	0,779	—	—
5,00	0,712	1,990	2,091	0,761	—	—	0,781	—	—	0,789	—	—
6,00	0,715	1,931	2,211	0,767	—	—	0,787	—	—	0,796	—	—
7,00	0,718	1,911	2,286	0,771	—	—	0,792	—	—	0,801	—	—
8,00	0,721	1,900	2,336	0,775	—	—	0,796	—	—	0,804	—	—
9,00	0,723	1,895	2,370	0,777	—	—	0,798	—	—	0,807	—	—
10,00	0,725	1,891	2,397	0,779	—	—	0,801	—	—	0,810	—	—
15,00	0,728	1,879	2,483	0,786	2,213	2,462	0,808	—	—	0.817	—	—
20,00	0,730	1,875	2,529	0,788	2,185	2,527	0,812	2,378	2,506	0,820	—	—
30,00	0,732	1,870	2,579	0,792	2,158	2,585	0,816	2,326	2,586	0,824	2,424	2,587
40,00	0,734	1,867	2,600	0,794	2,149	2,610	0,817	2,311	2,610	0,826	2,404	2,610
50,00	0,735	1 866	2,620	0,795	2,144	2,630	0,818	2,302	2,630	0,827	2,392	2,630

Table 38. Roots of characteristic equation (1.40) as a function of Pn ($Lu = 0{,}3$, $\varepsilon = 0{,}7$, $Bi_m = 20$, $Ko = 9$, $Lu_p = 500$)

Bi_q	Pn = 0,65		Pn = 0,45		Pn = 0,25			Pn = 0,05	
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_3	μ_1	μ_2
0,30	0,412	1,752	0,443	2,480	0,469	—	—	0,508	1,437
0,35	0,430	1,764	0,464	2,454	0,495	—	—	0,544	1,439
0,40	0,445	1,771	0,482	2,430	0,517	—	—	0,574	1,445
0,45	0,458	1,779	0,498	2,402	0,536	—	—	0,602	1,450
0,50	0,470	1,784	0,511	2,375	0,556	—	—	0,625	1,453
0,60	0,488	1,792	0,534	2,330	0,583	—	—	0,669	1,466
0,70	0,504	1,799	0,552	2,288	0,607	—	—	0,709	1,477
0,80	0,515	1,804	0,566	2,259	0,627	—	—	0,741	1,488
0,90	0,525	1,810	0,578	2,237	0,645	—	—	0,771	1,498
1,00	0,533	1,815	0,588	2,220	0,658	—	—	0,796	1,507
1,50	0,558	1,830	0,622	2,177	0,706	—	—	0,885	1,556
2,00	0,572	1,837	0,642	2,159	0,733	—	—	0,938	1,597
3,00	0,589	1,847	0,662	2,142	0,764	—	—	0,993	1,660
4,00	0,599	1,854	0,673	2,135	0,779	—	—	1,023	1,704
5,00	0,604	1,858	0,681	2,131	0,789	—	—	1,042	1,734
6,00	0,608	1,861	0,685	2,128	0,796	—	—	1,054	1,755
7,00	0,611	1,863	0,688	2.126	0,801	—	—	1,061	1,773
8,00	0,613	1,865	0,691	2.124	0,804	—	—	1,068	1,787
9,00	0,614	1,866	0,693	2,123	0,807	—	—	1,073	1,797
10,00	0,616	1,868	0,695	2,122	0,810	—	—	1,077	1,805
15,00	0,621	1,870	0,699	2,120	0,817	—	—	1,090	1,831
20.00	0,622	1,872	0,701	2,118	0,820	—	—	1,096	1,846
30,00	0,624	1,873	0,704	2,117	2,824	2,424	2,587	1,103	1,862
40.00	0,626	1,874	0,705	2,116	0,826	2,404	2,610	1,105	1,870
50,00	0,627	1,875	0,706	2,115	0,827	2,392	2,630	1,106	1,875

Table 39. Roots of characteristic equation (1.40) as a function of Ko ($Lu = 0{,}3$, $\varepsilon = 0{,}7$, $Bi_m = 20$, $Pn = 0{,}25$, $Lu_p = 500$)

Bi_q	Ko = 3		Ko = 6			Ko = 9			Ko = 12		
	μ_1	μ_2	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
0,30	0,502	1,489	0,485	1.686	2,191	0,469	—	—	0,456	—	—
0,35	0,535	1,498	0,513	1 699	2,189	0,495	—	—	0,480	—	—
0,40	0,565	1,508	0,538	1,720	2,187	0,517	—	—	0,501	—	—
0,45	0,590	1,518	0,563	1,743	2.185	0,536	—	—	0,518	—	—
0,50	0,614	1,527	0,582	1,768	2,182	0.556	—	—	0,534	—	—
0,60	0,656	1,546	0,617	1,816	2,179	0,583	—	—	0,560	—	—
0,70	0,690	1,563	0,646	1,862	2,173	0,607	—	—	0,581	—	—
0,80	0,718	1,579	0,670	1,904	2,163	0,627	—	—	0,599	—	—
0,90	0,743	1,595	0,690	1,948	2,148	0,645	—	—	0,613	—	—
1,00	0,766	1,612	0,707	1,988	2,129	0,658	—	—	0,625	—	—
1,50	0,843	1,687	0,765	—	—	0,706	—	—	0,666	—	—
2,00	0,886	1,734	0,798	—	—	0,733	—	—	0,690	—	—
3,00	0,934	1,807	0,836	—	—	0,764	—	—	0,716	—	—
4,00	0,961	1,852	0,856	—	—	0,779	—	—	0,730	—	—
5,00	0,975	1.884	0,867	—	—	0,789	—	—	0,738	—	—
6,00	0,987	1,907	0,875	2,340	2,422	0,796	—	—	0,744	—	—
7,00	0,995	1,924	0,881	2,317	2,503	0,801	—	—	0,748	—	—
8,00	1,000	1,937	4,886	2,307	2,547	0,804	—	—	0,751	2,415	2,575
9,00	1,005	1,946	0,889	2,302	2,569	0,807	—	—	0,754	2,386	2,603
10.00	1,008	1,956	0,891	2,296	2,605	0,810	—	—	0,756	2,358	2,639
15,00	1,020	1,978	0,900	2,287	2,671	0,817	—	—	0,761	2,328	2,700
20,00	1,025	1,991	0,904	2,282	2,710	0,820	—	—	0,764	2,315	2,725
30,00	1,030	2,003	0,908	2,279	2,747	0,824	2,424	2,587	0,766	2,303	2,747
40,00	1,033	2,007	0,911	2,277	2,768	0,826	2,404	2,610	0,769	2,299	2,759
50,00	1,035	2,012	0,913	2 276	2,777	0,827	2,392	2,630	0,770	2,295	2,764

Table 40. Roots of the characteristic equation (1.40) as a function of Lu_p ($Lu = 0{,}3$, $\varepsilon = 0{,}7$, $Bi_m = 20$, $KoPn = 2{,}25$)

Bi_q	$Lu_p = 300$			$Lu_p = 500$			$Lu_p = 1000$		
	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
0,30	0,454	1,677	2,260	0,469	—	—	0,468	—	—
0,35	0,480	1,700	2,246	0,495	—	—	0,493	—	—
0,40	0,503	1,726	2,238	0.517	—	—	0,517	—	—
0,45	0.523	1,754	2,232	0,536	—	—	0,537	—	—
0,50	0,541	1,778	2,228	0,556	—	—	0.554	—	—
0,60	0,572	1,836	2,222	0.583	—	—	0,584	—	—
0,70	0,596	1,889	2,214	0,607	—	—	0.607	—	—
0,80	0,617	1,942	2,199	0,627	—	—	0,628	—	—
0,90	0,634	2,005	2.173	0,645	—	—	0,644	—	—
1.00	0,648	—	—	0,658	—	—	0,658	—	—
1,50	0,699	—	—	0,706	—	—	0,707	—	—
2,00	0,727	—	—	0,733	—	—	0,734	—	—
3,00	0,758	—	—	0,764	—	—	0.765	—	—
4,00	0,775	—	—	0,779	—	—	0,781	—	—
5,00	0,786	—	—	0,789	—	—	0,791	—	—
6,00	0,793	—	—	0,796	—	—	0,799	—	—
7,00	0,798	—	—	0,801	—	—	0.803	—	—
8,00	0,802	—	—	0,804	—	—	0,807	—	—
9,00	0,805	—	—	0,807	—	—	0,810	—	—
10,00	0,808	—	—	0,810	—	—	0,813	—	—
15,00	0,816	—	—	0.817	—	—	0,820	—	—
20,00	0,819	—	—	0,820	—	—	0,824	—	—
30,00	0,823	—	—	0,824	2,424	2,587	0,828	2,428	2,581
40,00	0,825	—	—	0,826	2,404	2.610	0,830	2,404	2,616
50,00	0,826	—	—	0,827	2,392	2,630	0,831	2,394	2.635

Table 41. Roots of characteristic equation (1.40) as a function of

ε ($\mathrm{Lu} = 0{,}3$, $\mathrm{Lu}_p = 500$, $\mathrm{Bi}_m = 20$, $\mathrm{Pn} = 0{,}25$, $\mathrm{Ko} = 9$)

Bi_q	$\varepsilon = 0{,}1$			$\varepsilon = 0{,}7$			$\varepsilon = 0{,}9$	
	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3	μ_1	μ_2
0,30	0,463	1,345	1.959	0,469	—	—	0,474	—
0.35	0,490	1,351	1,961	0,495	—	—	0,501	—
0,40	0,512	1,358	1,967	0,517	—	—	0,523	—
0,45	0,533	1,369	1,970	0.536	—	—	0,543	—
0,50	0,550	1,381	1,978	0,556	—	—	0,561	—
0.60	0,579	1,405	1,993	0,583	—	—	0,590	—
0,70	0,603	1 422	2,012	0,607	—	—	0,614	—
0,80	0 623	1,438	2,030	0,627	—	—	0,633	—
0,90	0,639	1,453	2,047	0,645	—	—	0,649	—
1,00	0,653	1,466	2,062	0,658	—	—	0,662	—
1,50	0,702	1,510	2,123	0,706	—	—	0,709	—
2,00	0,731	1,536	2,178	0,733	—	—	0,736	—
3,00	0,761	1,568	2,247	0,764	—	—	0,767	—
4,00	0,776	1,586	2,286	0,779	—	—	0,784	2,616
5,00	0,785	1,598	2,313	0,789	—	—	0,794	2,560
6,00	0,792	1,607	2,334	0,796	—	—	0,803	2,537
7,00	0,796	1,613	2,352	0,801	—	—	0,807	2,523
8,00	0,800	1,617	2,364	0.804	—	—	0,812	2,514
9,00	0,802	1,620	2,376	0,807	—	—	0,815	2,507
10,00	0,804	1,623	2,384	0,810	—	—	0,817	2,502
15,00	0,810	1,631	2,413	0,817	—	—	0.825	2.490
20,00	0,813	1,636	2,427	0,820	—	—	0,828	2.484
30,00	0,816	1,640	2,442	0,824	2,424	2,587	0,832	2,478
40,00	0,818	1 642	2,450	0,826	2.404	2,610	0,834	2,475
50,00	0,819	1,644	2,455	0.827	2,392	2.630	0,836	2,473

Table 42. Roots of characteristic equation of (1.40) as function of Lu ($\varepsilon = 0{,}7$, $\mathrm{Bi}_m = 20$, $\mathrm{Ko} = 9$, $\mathrm{Pn} = 0{,}25$, $\mathrm{Lu}_p = 500$)

Bi_q	Lu = 0,3			Lu = 0,7		Lu = 1,0	Lu = 3,333
	μ_1	μ_2	μ_3	μ_1	μ_2	μ_1	μ_1
0,25	0,438	—	—	0,0714	1,970	0,0529	0,0435
0,30	0,469	—	—	0,0779	1,971	0,0578	0,0474
0,35	0,495	—	—	0,0834	1,972	0,0622	0,0508
0,40	0,517	—	—	0,0884	1,974	0,0662	0,0541
0,45	0,536	—	—	0,0928	1,975	0,0699	0,0571
0,50	0,556	—	—	0,0968	1,977	0,0733	0,0598
0 60	0,583	—	—	0,1050	1,978	0,0801	0,0652
0,70	0,607	—	—	0,111	1,980	0,0858	0,0698
0,80	0,627	—	—	0,117	1,984	0,0911	0,0738
0,90	0,645	—	—	0,123	1,986	0,0962	0,0778
1,00	0,658	—	—	0,128	1,987	0,1011	0,0815
1 50	0,706	—	—	0,151	1,998	0,1197	0,0950
2,00	0,733	—	—	0,170	2,008	0,1340	0,106
3,00	0,764	—	—	0,197	2,024	0,157	0,124
4,00	0,779	—	—	0,214	2,037	0,173	0,138
5,00	0,789	—	—	0,228	2,047	0,184	0,155
6,00	0,796	—	—	0,242	2,056	0,197	0,167
7,00	0,801	—	—	0,250	2,064	0,208	0,175
8,00	0,804	—	—	0,254	2,069	0,214	0,188
9,00	0,807	—	—	0,260	2,074	0,220	—
10,00	0,810	—	—	0,263	2,078	0,225	0,204
15.00	0,817	—	—	0,280	2,091	0,245	0,231
20.00	0,820	—	—	0,287	2,098	0,256	0,241
30.00	0,824	2,424	2,587	0,296	2,108	0,269	0,255
40,00	0,826	2,404	2,610	0,300	2,112	0,275	0,264
50,00	0,827	2,392	2,630	0,302	2,114	0,278	0,270

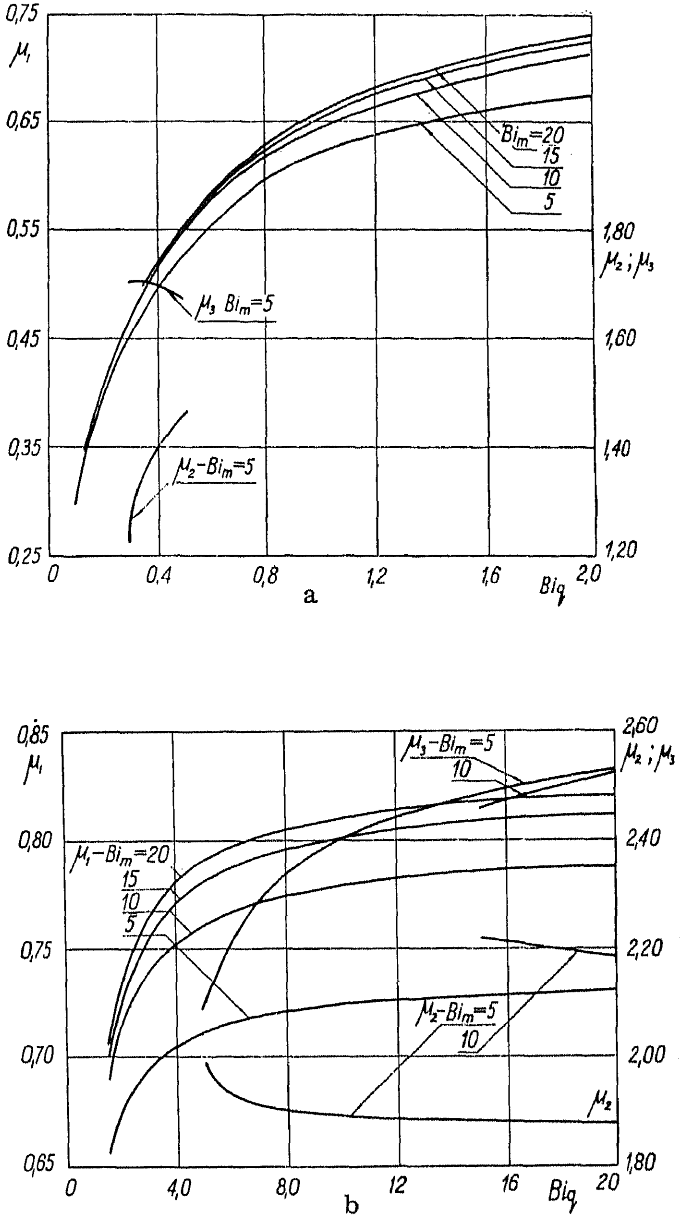
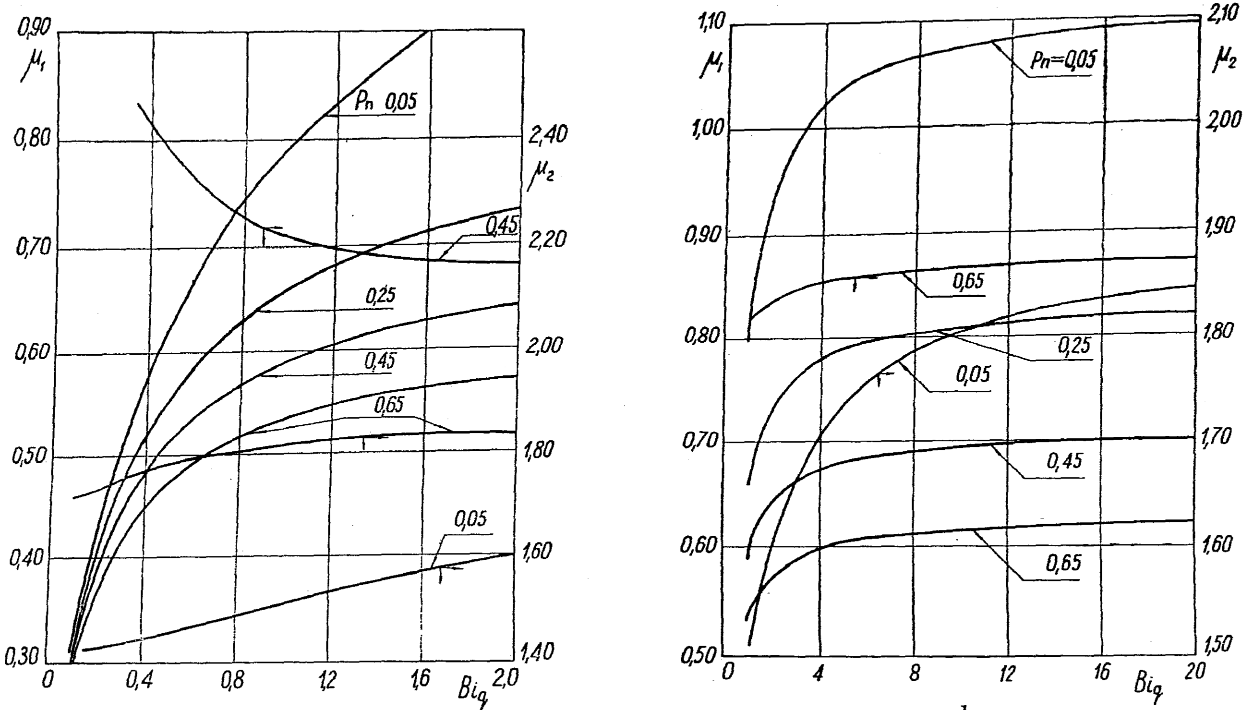


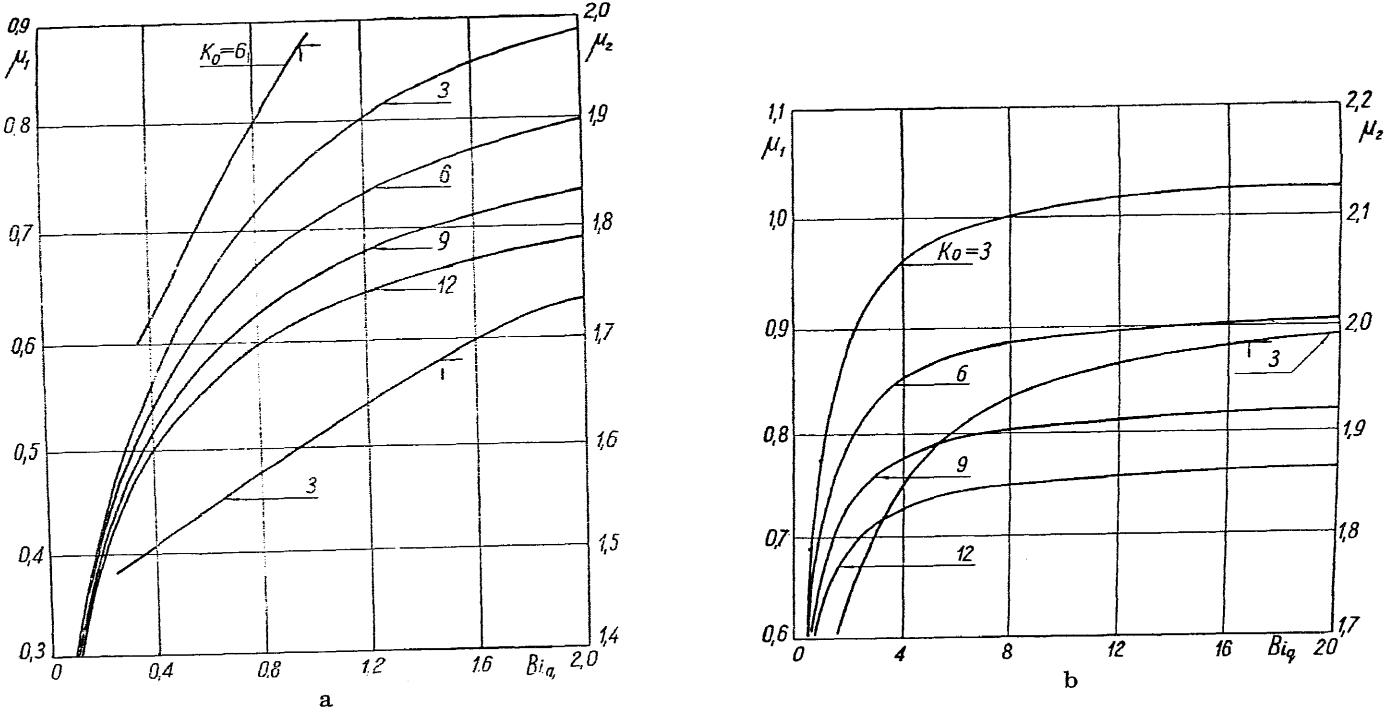
Fig. 48. Characteristic roots for an infinite plate as a function of:
a — $Bi_m(Bi_q = 0.1\text{–}2.0)$, b — $Bi_m(Bi_q = 1.0\text{–}20)$.



a

b

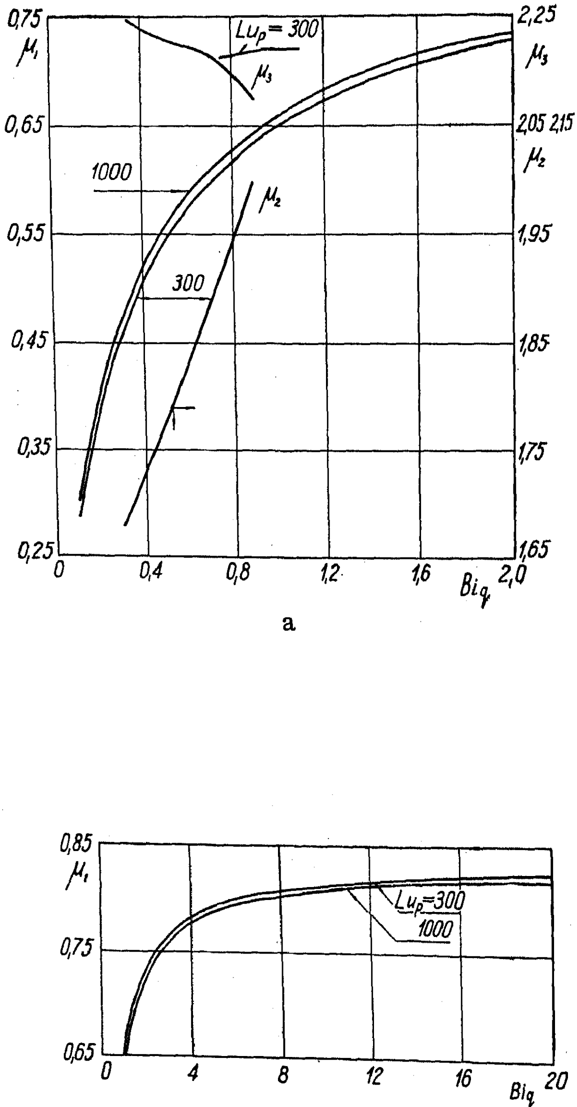
Fig. 49. Characteristic roots for an infinite plate as a function of:
a — Pn($Bi_q = 0.1$-2.0), b — Pn($Bi_q = 1.0$-20).



a

b

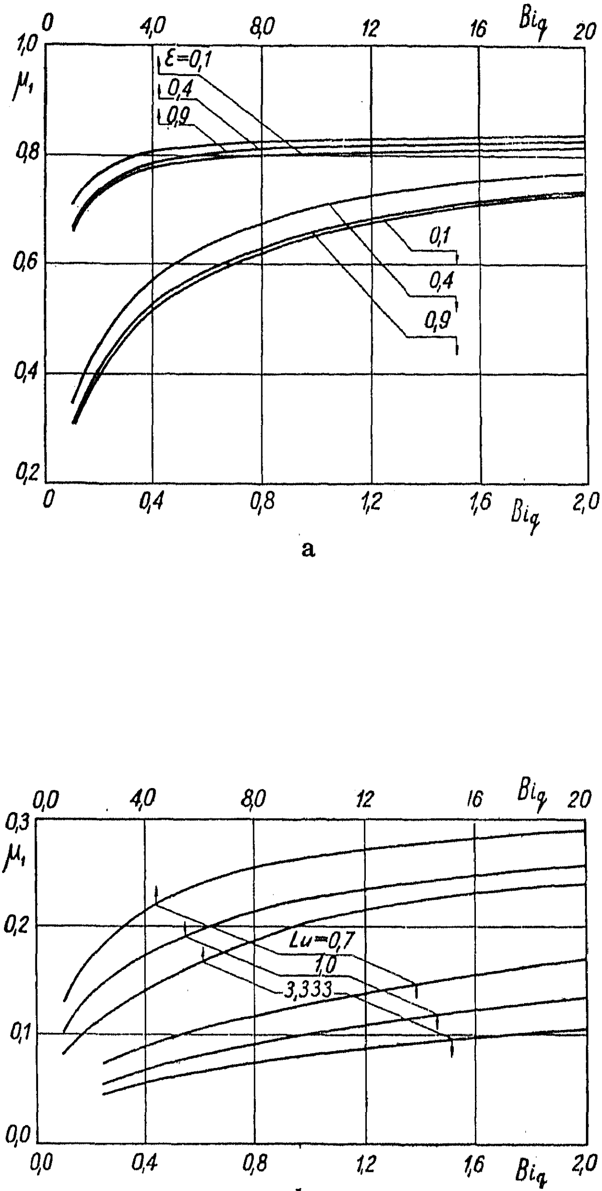
Fig. 50. Characteristic roots of an infinite plate as a function of a — Ko($Bi_q = 0.1-2.0$), b — Ko($Bi_q = 1.0-20$).



a

b

Fig. 51. Characteristic roots for an infinite plate as a function of
a — $Lu_p(Bi_q = 0.1\text{-}2.0)$, b — $Lu_p(Bi_q = 1.0\text{-}20)$.



a

b

Fig. 52. Characteristic roots of an infinite plate as a function of
a — ε, b — Lu.

The average value of a dimensionless quantity is given by the following formula:

$$\overline{\varphi}(\mathrm{Fo}) = \int_0^1 \varphi(X, \mathrm{Fo})\, dX,$$

therefore, for average dimensionless transfer potentials, we get

$$\overline{T} = 1 - \sum_{n=1}^{\infty} \sum_{j=1}^{3} D_{nj} \exp\left(-\mu_n^2 \mathrm{Fo}\right), \tag{1.41}$$

$$\overline{\theta} = 1 + \frac{1}{\varepsilon \mathrm{Ko}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} D_{nj} \left(1 - \nu_j^2\right) \exp\left(-\mu_n^2 \mathrm{Fo}\right), \tag{1.42}$$

$$\overline{P} = -\frac{1}{\varepsilon \mathrm{Bu}} \sum_{n=1}^{\infty} \sum_{j=1}^{3} D_{nj} \sigma_j \exp\left(-\mu_n^2 \mathrm{Fo}\right). \tag{1.43}$$

Here,

$$D_{nj} = C_{nj} \frac{\sin \nu_j \mu_n}{\nu_j \mu_n}.$$

The expressions for rates of change of the corresponding parameters and for consumption of heat and mass are easily obtained using the well-known formulae given above.

Results of numerical calculations in accordance with formulae (1.27) to (1.29) and (1.41) to (1.43) show that the infinite series appearing in the solutions converge quite rapidly. Beginning with a certain value of the Fourier number only the first two or three terms have to be utilized for the computation. On the average, beginning with the value of $\mathrm{Fo} = 0.7$, the error committed by using only the first two or three terms of the series does not exceed 1-2%. On this basis, equations (1.41) to (1.43) can be simplified to the following form for practical computations:

$$\overline{T} \approx 1 - \sum_{n=1}^{2} D_{n\mathrm{T}} \exp\left(-\mu_n^2 \mathrm{Fo}\right), \tag{1.44}$$

$$\overline{\Theta} \approx 1 + \sum_{n=1}^{2} D_{n\Theta} \exp\left(-\mu_n^2 \mathrm{Fo}\right), \tag{1.45}$$

$$\overline{P} \approx \sum_{n=1}^{2} D_{np} \exp\left(-\mu_n^2 \mathrm{Fo}\right), \tag{1.46}$$

where

$$D_{n\mathrm{T}} = \sum_{j=1}^{3} D_{nj}, \qquad D_{n\theta} = \frac{1}{\varepsilon\,\mathrm{Ko}} \sum_{j=1}^{3} D_{nj}\,(1-\nu_j^2),$$

$$D_{np} = -\frac{1}{\varepsilon\,\mathrm{Bu}} \sum_{j=1}^{3} D_{nj}\,\sigma_j.$$

The values of coefficients C_{nj}, D_{nj}, and $D_{n\mathrm{T}}$, $D_{n\theta}$, D_{np} for various dimensionless numbers of heat and mass transfer are given in Tables 43–47.

b) Sphere.

The system of differential equations for high intensity heat and mass transfer (XVIII) in a sphere can be put into the following form:

$$\frac{\partial[XT]}{\partial\mathrm{Fo}} = \frac{\partial^2[XT]}{\partial X^2} - \varepsilon\,\mathrm{Ko}\,\frac{\partial[X\Theta]}{\partial\mathrm{Fo}}, \tag{1.47}$$

$$\frac{\partial[X\Theta]}{\partial\mathrm{Fo}} = \mathrm{Lu}\,\frac{\partial^2[X\Theta]}{\partial X^2} - \mathrm{Lu}\,\mathrm{Pn}\,\frac{\partial^2[XT]}{\partial X^2} - \mathrm{Lu}_p\,\frac{\mathrm{Bu}}{\mathrm{Ko}}\,\frac{\partial^2[XP]}{\partial X^2} \tag{1.48}$$

$$\frac{\partial[XP]}{\partial\mathrm{Fo}} = \mathrm{Lu}_p\,\frac{\partial^2[XP]}{\partial X^2} + \varepsilon\,\frac{\mathrm{Ko}}{\mathrm{Bu}}\,\frac{\partial[X\Theta]}{\partial\mathrm{Fo}}. \tag{1.49}$$

Here, we denote

$$XT = T^*, \qquad X\Theta = \Theta^*, \qquad XP = P^* \tag{1.50}$$

Thus, the notation here is identical to that used for an infinite plate. Consequently the transformed solution is

$$\begin{aligned} T^*_L &= A_1 \exp(\sqrt{s}\,\nu_1 X) + A_2 \exp(\sqrt{s}\,\nu_2 X) + A_3 \exp(\sqrt{s}\,\nu_3 X) + \\ &+ A_4 \exp(-\sqrt{s}\,\nu_1 X) + A_5 \exp(-\sqrt{s}\,\nu_2 X) + \\ &+ A_6 \exp(-\sqrt{s}\,\nu_3 X), \end{aligned} \tag{1.51}$$

$$\begin{aligned}\Theta_L^* = &-\frac{1}{\varepsilon\,\mathrm{Ko}}[A_1(1-\nu_1^2)\exp(\sqrt{s}\,\nu_1 X)+A_2(1-\nu_2^2)\exp(\sqrt{s}\,\nu_2 X)+\\ &+A_3(1-\nu_3^2)\exp(\sqrt{s}\,\nu_3 X)+A_4(1-\nu_1^2)\exp(-\sqrt{s}\,\nu_1 X)+\\ &+A_5(1-\nu_2^2)\exp(-\sqrt{s}\,\nu_2 X)+A_6(1-\nu_3^2)\exp(-\sqrt{s}\,\nu_3 X)],\end{aligned} \tag{1.52}$$

$$\begin{aligned}P_L^* = &\frac{1}{\varepsilon\,\mathrm{Bu}}[A_1\sigma_1\exp(\sqrt{s}\,\nu_1 X)+A_2\sigma_2\exp(\sqrt{s}\,\nu_2 X)+\\ &+A_3\sigma_3\exp(\sqrt{s}\,\nu_3 X)+A_4\sigma_1\exp(-\sqrt{s}\,\nu_1 X)+\\ &+A_5\sigma_2\exp(-\sqrt{s}\,\nu_2 X)+A_6\sigma_3\exp(-\sqrt{s}\,\nu_3 X)],\end{aligned} \tag{1.53}$$

where σ_j and ν_j are determined by equations (1.13) and (1.14), respectively.

The conditions of symmetry for a sphere (1.14) transform to

$$T_L'(0,s) = \Theta_L'(0,s) = P_L'(0,s) = 0. \tag{1.54}$$

In order that equations (1.51) to (1.53) satisfy the conditions (1.54) we must have

$$A_1 = -A_4, \quad A_2 = -A_5, \quad A_3 = -A_6.$$

Consequently, the solution of the problem has the following form:

$$T_L^*(X,s) = \sum_{j=1}^{3} C_j^* \,\mathrm{sh}\,\sqrt{s}\,\nu_j X, \tag{1.55}$$

$$\Theta_L^*(X,s) = -\frac{1}{\varepsilon\,\mathrm{Ko}}\sum_{j=1}^{3} C_j^*(1-\nu_j^2)\,\mathrm{sh}\,\sqrt{s}\,\nu_j X, \tag{1.56}$$

$$P_L^*(X,s) = \frac{1}{\varepsilon\,\mathrm{Bu}}\sum_{j=1}^{3} C_j^*\sigma_j\,\mathrm{sh}\,\sqrt{s}\,\nu_j X. \tag{1.57}$$

Substituting equations (1.55) to (1.57) into boundary conditions (1.18) to (1.20) and considering equation (1.50) we get the following system of algebraic equations:

$$\left.\begin{array}{l}\displaystyle\sum_{j=1}^{3} C_j^* P_j^* + \frac{\varepsilon\,\mathrm{Ko}}{s} = 0\\ \displaystyle\sum_{j=1}^{3} C_j^* Q_j^* + \frac{\varepsilon\,\mathrm{Ko}K_1 - 1}{s} = 0\\ \displaystyle\sum_{j=1}^{3} C_j^* \chi_j^* = 0.\end{array}\right\}, \tag{1.58}$$

Bi_q

Table 43. Values of constant coefficients for an infinite plate

($\mathrm{Lu} = 0{,}3$, $\varepsilon = 0{,}7$, $\mathrm{Bi}_m = 20$, $\mathrm{Ko} = 9$, $\mathrm{Pn} = 0{,}25$, $\mathrm{Lu}_p = 500$, $\mathrm{Bu} = 1 \cdot 10^{-3}$)

	$\mathrm{Bi}_q = 5$	$\mathrm{Bi}_q = 10$	$\mathrm{Bi}_q = 15$	$\mathrm{Bi}_q = 20$	$\mathrm{Bi}_q = 30$		
	μ_1	μ_1	μ_1	μ_1	μ_1	μ_2	μ_3
C_1	3,053	3,052	3,052	3,054	3,051	−1.586	0,6413
C_2	$-0.6316 \cdot 10^{-2}$	$-0{,}1760 \cdot 10^{-2}$	$-0{,}1822 \cdot 10^{-3}$	$0{,}6397 \cdot 10^{-3}$	$0{,}1437 \cdot 10^{-2}$	$0{,}4536 \cdot 10^{-2}$	$0{,}1186 \cdot 10^{-2}$
C_3	0,7861	0,2965	0.1252	$0{,}3672 \cdot 10^{-1}$	−0,05070	−0,9590	−0,4161
D_1	2,118	2,073	2,057	2,051	2,041	+0,3424	−0,1370
D_2	$-0{,}6311 \cdot 10^{-2}$	$-0{,}1759 \cdot 10^{-2}$	$-0{,}1821 \cdot 10^{-3}$	$0{,}6393 \cdot 10^{-3}$	$0{,}1436 \cdot 10^{-2}$	$0{,}4507 \cdot 10^{-2}$	$0{,}1177 \cdot 10^{-2}$
D_3	0,7617	0,2868	0,1210	$0{,}3549 \cdot 10^{-1}$	−0,04899	−0,6992	−0,2894
D_{nT}	2,873	2,358	2,178	2,087	1,993	−0,3523	−0,4252
$D_{n\theta}$	−0,6792	−0,7149	−0,7271	−0,7344	−0,7400	−0,2000	0,01746
D_{np}	2,420	2,657	2,746	2,790	2,834	1,910	0,6403

Bi_m

Table 44. Values of constant coefficients for an infinite plate

($\mathrm{Lu} = 0{,}3$, $\varepsilon = 0{,}7$, $\mathrm{Bi}_q = 20$, $\mathrm{Ko} = 9$, $\mathrm{Pn} = 0{,}25$, $\mathrm{Lu}_p = 500$, $\mathrm{Bu} = 1 \cdot 10^{-3}$)

	$\mathrm{Bi}_m = 5$			$\mathrm{Bi}_m = 8$		
	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
C_1	3,022	—0,6713	—0,1646	3,032	—0,8157	$-0{,}3260 \cdot 10^{-1}$
C_2	$0{,}8495 \cdot 10^{-2}$	$0{,}9549 \cdot 10^{-2}$	$-0{,}2836 \cdot 10^{-3}$	$0{,}4320 \cdot 10^{-2}$	$0{,}8902 \cdot 10^{-2}$	$-0{,}1388 \cdot 10^{-4}$
C_3	—0,5517	—1,221	0,1844	—0,2480	—1,337	$0{,}4945 \cdot 10^{-1}$
D_1	2,218	$0{,}4870 \cdot 10^{-1}$	$0{,}3565 \cdot 10^{-1}$	2,120	0,1233	$-0{,}7061 \cdot 10^{-2}$
D_2	$0{,}8489 \cdot 10^{-2}$	$0{,}9512 \cdot 10^{-2}$	$-0\ 2816 \cdot 10^{-3}$	$0{,}4317 \cdot 10^{-2}$	$0{,}8660 \cdot 10^{-2}$	$-0{,}1378 \cdot 10^{-4}$
D_3	—0,5370	—1,016	0 1305	—0,2404	—1,067	$0{,}3500 \cdot 10^{-1}$
D_{nT}	1,689	—0,958	0,1659	1,884	—0,935	$0{,}2792 \cdot 10^{-1}$
$D_{n\theta}$	—0,8565	—0,1286	$0{,}1576 \cdot 10^{-2}$	—0,7892	—0,1611	$0{,}6416 \cdot 10^{-2}$
D_{np}	2,437	0,8314	—0,4340	2,646	1,459	0,1683

Table 44, continued

	$\mathrm{Bi}_m = 15$			$\mathrm{Bi}_m = 20$
	μ_1	μ_2	μ_3	μ_1
C_1	3,049	—1,977	0,9574	3,054
C_2	$0,1445 \cdot 10^{-2}$	$0,6335 \cdot 10^{-3}$	$0,5966\ 10^{-3}$	$0,6397 \cdot 10^{-3}$
C_3	$-0,2636 \cdot 10^{-1}$	—1,114	—0,5776	$0,3672 \cdot 10^{-1}$
D_1	2,066	0,4211	—0,2079	2,051
D_2	$0,1444 \cdot 10^{-2}$	$0,6296 \cdot 10^{-2}$	$0,5925 \cdot 10^{-3}$	$0,6393 \cdot 10^{-3}$
D_3	$-0,2549 \cdot 10^{-1}$	—0,8228	—0,4114	$0,3549 \cdot 10^{-1}$
D_{nm}	2,042	—0,3954	—0,6187	2,087
$D_{n\theta}$	—0,7463	—0,2417	$0,2937 \cdot 10^{-1}$	—0,7344
D_{np}	2.761	1,860	1,341	2,790

Pn

Table 45. Values of constant coefficients for an infinite plate

($\mathrm{Lu} = 0{,}3$, $\varepsilon = 0{,}7$, $\mathrm{Bi}_q = 20$, $\mathrm{Bi}_m = 20$, $\mathrm{Ko} = 9$, $\mathrm{Lu}_p = 500$, $\mathrm{Bu} = 1 \cdot 10^{-3}$)

	Pn = 0,6500		Pn = 0,4500		Pn = 0,3333		
	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2	μ_3
C_1	1,576	$-0{,}5375$	2,063	$-0{,}7563$	2,528	$-1{,}138$	0,3658
C_2	$-0{,}1663 \cdot 10^{-2}$	$-0{,}1889 \cdot 10^{-2}$	$-0{,}8842 \cdot 10^{-3}$	$-0{,}1420 \cdot 10^{-2}$	$-0{,}2165 \cdot 10^{-3}$	$-0{,}1152 \cdot 10^{-2}$	$0{,}5173 \cdot 10^{-2}$
C_3	$0{,}8207 \cdot 10^{-1}$	0,1186	$0{,}8207 \cdot 10^{-1}$	0,1857	$0{,}7269 \cdot 10^{-1}$	0,3447	$-1{,}579$
D_1	1,039	0,1165	1,367	0,1640	1,684	0,2469	$-0{,}5557 \cdot 10^{-1}$
D_2	$-0{,}1662 \cdot 10^{-2}$	$-0{,}1881 \cdot 10^{-2}$	$-0{,}8838 \cdot 10^{-3}$	$-0{,}1413 \cdot 10^{-2}$	$-0{,}2165 \cdot 10^{-3}$	$-0{,}1145 \cdot 10^{-2}$	$0{,}5131 \cdot 10^{-2}$
D_3	$0{,}8119 \cdot 10^{-1}$	0,1075	$0{,}8065 \cdot 10^{-1}$	0,1576	$0{,}7089 \cdot 10^{-1}$	0,2709	$-1{,}123$
D_{nT}	1,119	0,2221	1,447	0,3202	1,755	0,5167	$-1{,}174$
$D_{n\theta}$	$-0{,}8017$	$-0{,}7711 \cdot 10^{-1}$	$-0{,}7751$	$-0{,}7471 \cdot 10^{-1}$	$-0{,}7506$	$-0{,}7946 \cdot 10^{-1}$	$-0{,}1067$
D_{np}	2,017	$-0{,}1587$	1,919	0,3226	2,317	$-0{,}7329$	4,400

Table 45, continued

	Pn = 0,2500	Pn = 0 1667			Pn = 0,0500	
	μ_1	μ_1	μ_2	μ_3	μ_1	μ_2
C_1	3,054	3,927	−0,5818	−0,6727	7,849	−0,8482
C_2	$0{,}6397\cdot10^{-3}$	$0{,}1025\cdot10^{-2}$	$0{,}5297\cdot10^{-2}$	$0{,}5673\cdot10^{-3}$	$0{,}3174\cdot10^{-2}$	$0{,}4937\cdot10^{-2}$
C_3	$0{,}3672\cdot10^{-1}$	$0{,}6170\cdot10^{-2}$	−2,532	$0{,}1027\cdot10^{-1}$	−0.4750	−5,725
D_1	2,051	2,644	$0{,}8686\cdot10^{-1}$	0,1458	5,386	−0,2262
D_2	$0{,}6393\cdot10^{-3}$	$0{,}1024\cdot10^{-2}$	$0{,}5266\cdot10^{-2}$	$0{,}5627\cdot10^{-3}$	$0{,}3170\cdot10^{-2}$	0,4917
D_3	$0{,}3549\cdot10^{-1}$	$0{,}5864\cdot10^{-2}$	−1,794	$0{,}6208\cdot10^{-2}$	−0,4225	−4,038
D_{nT}	2,087	2,651	−1,702	0,1526	4,967	−4,259
$D_{n\theta}$	−0,7344	−0,7022	−0,2016	$-0{,}3805\cdot10^{-1}$	−0,6598	−0,2473
D_{np}	2,790	2,836	4,074	$-0{,}7657\cdot10^{-1}$	3,904	5,847

Ko

Table 46. Values of constant coefficients for an infinite plate

($Lu = 0{,}3$, $\varepsilon = 0{,}7$, $Bi_q = 20$, $Bi_m = 20$, $Pn = 0{,}25$, $Lu_p = 500$, $Bu = 1 \cdot 10^{-3}$)

	Ko = 3		Ko = 6		
	μ_1	μ_2	μ_1	μ_2	μ_3
C_1	2,199	—0,1131	2,721	—0,3165	—0,5500
C_2	$0{,}8457 \cdot 10^{-2}$	$0{,}1013 \cdot 10^{-3}$	$0{,}7109 \cdot 10^{-3}$	$0{,}2889 \cdot 10^{-2}$	$0{,}4604 \cdot 10^{-3}$
C_3	—0,067772	—0,8361	$0{,}4196 \cdot 10^{-2}$	—1,383	$0{,}3319 \cdot 10^{-2}$
D_1	1,407	$-0{,}1195 \cdot 10^{-1}$	1,832	$0{,}4725 \cdot 10^{-1}$	0,1192
D_2	$0{,}8447 \cdot 10^{-3}$	$0{,}1009 \cdot 10^{-2}$	$0{,}7103 \cdot 10^{-3}$	$0{,}2872 \cdot 10^{-2}$	$0{,}4567 \cdot 10^{-3}$
D_3	—0,0608	—0,5911	$0{,}3988 \cdot 10^{-2}$	—0,9797	$0{,}2006 \cdot 10^{-2}$
$D_{n\mathrm{T}}$	1,436	—0,6020	1,835	—0,9296	0,1217
$D_{n\theta}$	—0,7505	—0,1376	—0,7305	—0,1651	$-0{,}1733 \cdot 10^{-1}$
D_{np}	1,313	0,8053	1,979	2,226	$-0{,}5047 \cdot 10^{-1}$

Table 46, continued

	Ko = 9	Ko = 12		
	μ_1	μ_1	μ_2	μ_3
C_1	3,054	3,291	—1,557	0,5701
C_2	$0{,}6397 \cdot 10^{-3}$	$-0{,}2814 \cdot 10^{-3}$	$-0{,}1577 \cdot 10^{-2}$	$0{,}8058 \cdot 10^{-2}$
C_3	$0{,}3672 \cdot 10^{-1}$	$0{,}9119 \cdot 10^{-1}$	0,4718	—2,460
D_1	2,051	2,192	0,3379	$-0{,}8660 \cdot 10^{-1}$
D_2	$0{,}6393 \cdot 10^{-3}$	$-0{,}2812 \cdot 10^{-3}$	$-0{,}1568 \cdot 10^{-2}$	$0{,}7993 \cdot 10^{-2}$
D_3	$0{,}3549 \cdot 10^{-1}$	$0{,}8893 \cdot 10^{-1}$	0,3708	—1,750
D_{nT}	2,087	2,281	0,7071	—1,829
$D_{n\Theta}$	—0,7344	—0,7330	$-0{,}8157 \cdot 10^{-1}$	—0,1246
D_{np}	2,790	3,037	—0,9740	6,859

Lu_p

Table 47. Values of constant coefficients for an infinite plate

($\mathrm{Lu} = 0{,}3$, $\varepsilon = 0{,}7$, $\mathrm{Bi}_q = 20$, $\mathrm{Bi}_m = 20$, $\mathrm{Bu} = 1 \cdot 10^{-3}$, $\mathrm{Ko} = 9$, $\mathrm{Pn} = 0{,}25$)

	$\mathrm{Lu}_p = 300$	$\mathrm{Lu}_p = 500$	$\mathrm{Lu}_p = 700$	$\mathrm{Lu}_p = 1000$
	μ_1	μ_1	μ_1	μ_1
C_1	3,031	3,054	3,073	3,040
C_2	$0{,}1588 \cdot 10^{-2}$	$0{,}6397 \cdot 10^{-3}$	$0{,}2718 \cdot 10^{-3}$	$0\ 1887 \cdot 10^{-3}$
C_3	$0{,}3014 \cdot 10^{-1}$	$0{,}3672 \cdot 10^{-1}$	$0{,}4202 \cdot 10^{-1}$	$0{,}5231 \cdot 10^{-1}$
D_1	2,038	2,051	2,063	2,033
D_2	$0{,}1587 \cdot 10^{-2}$	$0{,}6393 \cdot 10^{-3}$	$0{,}2717 \cdot 10^{-3}$	$0{,}1886 \cdot 10^{-3}$
D_3	$0{,}2913 \cdot 10^{-1}$	$0{,}3549 \cdot 10^{-1}$	$0{,}4061 \cdot 10^{-3}$	$0{,}5053 \cdot 10^{-1}$
$D_{n\mathrm{T}}$	2,069	2,087	2,104	2,084
$D_{n\theta}$	—0,7298	—0,7344	—0,7381	—0,7267
D_{np}	4,931	2,790	1,609	1,371

where

$$Q_j^* = \left[1 - \frac{1}{\mathrm{Bi}_q} + \left(1 - \nu_j^2\right) K_1 \right] \mathrm{sh} \sqrt{s}\, \nu_j +$$

$$+ \frac{1}{Bi_q} \sqrt{s}\, \nu_j \,\mathrm{ch} \sqrt{s}\, \nu_j,$$

$$\mathrm{P}_j^* = \frac{(1-\nu_j^2) + \varepsilon \,\mathrm{Ko}\, \mathrm{Pn} + \dfrac{\mathrm{Lu}_p}{\mathrm{Lu}} \sigma_j}{\mathrm{Bi}_m} (\sqrt{s}\, \nu_j \,\mathrm{ch} \sqrt{s}\, \nu_j - \mathrm{sh} \sqrt{s}\, \nu_j) +$$

$$+ (1 - \nu_j^2) \,\mathrm{sh} \sqrt{s}\, \nu_j,$$

$$\chi_j^* = \sigma_j \,\mathrm{sh} \sqrt{s}\, \nu_j.$$

This solution may be written as

$$C_j^* = \left[(\varepsilon \,\mathrm{Ko}\, K_1 - 1) \left(P_2^* - \frac{\chi_2^*}{\chi_3^*} P_3^* \right) - \varepsilon \,\mathrm{Ko} \left(Q_2^* - \frac{\chi_2^*}{\chi_3^*} Q_3^* \right) \right] \times$$

$$\times \left\{ s \left[\left(P_1^* - \frac{\chi_1^*}{\chi_3^*} \right) \left(Q_2^* - \frac{\chi_2^*}{\chi_3^*} Q_3^* \right) - \right.\right.$$

$$\left.\left. - \left(P_2^* - \frac{\chi_2^*}{\chi_3^*} P_3^* \right) \left(Q_1^* - \frac{\chi_1^*}{\chi_3^*} Q_3^* \right) \right] \right\}^{-1},$$

$$C_2^* = - \left[(\varepsilon \,\mathrm{Ko} K_1 - 1) \left(P_1^* - \frac{\chi_1^*}{\chi_3^*} P_3^* \right) - \varepsilon \,\mathrm{Ko} \left(Q_1^* - \frac{\chi_1^*}{\chi_3^*} Q_3^* \right) \right] \times$$

$$\times \left\{ s \left[\left(P_1^* - \frac{\chi_1^*}{\chi_3^*} P_3^* \right) \left(Q_2^* - \frac{\chi_2^*}{\chi_3^*} Q^* \right) - \right.\right.$$

$$\left.\left. - \left(P_2^* - \frac{\chi_2^*}{\chi_3^*} P_3^* \right) \left(Q_1^* - \frac{\chi_1^*}{\chi_3^*} Q_3^* \right) \right] \right\}^{-},$$

$$C_3^* = \left[(\varepsilon \mathrm{Ko} K_1 - 1) \left(P_1^* \frac{\chi_2^*}{\chi_3^*} - P_2^* \frac{\chi_1^*}{\chi_3^*} \right) - \varepsilon \mathrm{Ko} \left(Q_1^* \frac{\chi_2^*}{\chi_3^*} - Q_2^* \frac{\chi_1^*}{\chi_3^*} \right) \right] \times$$

$$\times \left\{ s \left[\left(P_1^* - \frac{\chi_1^*}{\chi_3^*} P_3^* \right) \left(Q_2^* - \frac{\chi_2^*}{\chi_3^*} Q_3^* \right) - \right.\right.$$

$$-\left(P_2^* - \frac{\chi_2^*}{\chi_3^*}P_3^*\right)\left(Q_1^* - \frac{\chi_1^*}{\chi_3^*}Q_3^*\right)\Big]\Big\}^{-1}.$$

Thus, the transform of the problem under consideration has been solved.

The inversion of (1.55) to (1.57) is obtained from the expansion theorem, after converting the numerator and denominator of corresponding expressions into generalized polynomials. Because the method of transforming generalized polynomials for a sphere has been considered many times in the previous chapters, we give only the final solution:

$$T^* = X + \sum_{n=1}^{\infty}\sum_{j=1}^{3} C_{nj} \sin \nu_j \mu_n X \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.59}$$

$$\theta^* = X - \frac{1}{\varepsilon \mathrm{Ko}} \sum_{n=1}^{\infty}\sum_{j=1}^{3} C_{nj}(1 - \nu_j^2) \sin \nu_j \mu_n \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.60}$$

$$P^* = \frac{1}{\varepsilon \mathrm{Bu}} \sum_{n=1}^{\infty}\sum_{j=1}^{3} C_{nj}\, \sigma_j \sin \nu_j \mu_n X \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.61}$$

or, after equation (1.50) is taken into account,

$$\mathrm{T} = 1 + \sum_{n=1}^{\infty}\sum_{j=1}^{3} C_{nj} \frac{\sin \nu_j \mu_n X}{X} \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.59'}$$

$$\theta = 1 - \frac{1}{\varepsilon \mathrm{Ko}} \sum_{n=1}^{\infty}\sum_{j=1}^{3} C_{nj}(1 - \nu_j^2) \frac{\sin \nu_j \mu_n X}{X} \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.60'}$$

$$P = \frac{1}{\varepsilon \mathrm{Bu}} \sum_{n=1}^{\infty}\sum_{j=1}^{3} C_{nj}\, \sigma_j \frac{\sin \nu_j \mu_n X}{X} \exp(-\mu_n^2 \mathrm{Fo}), \tag{1.61'}$$

here, C_{nj} and ψ_n are determined by equations (1.30) to (1.33), respectively. The characteristic roots μ_n are determined by equation (1.40),

$$A_{nj} = [1 + (1 - \nu_j^2)K_1] \cos \nu_j \mu_n - \frac{1}{\mathrm{Bi}_q} \nu_j \mu_n \sin \nu_j \mu_n,$$

$$B_{nj} = (1 - \nu_j^2)\cos \nu_j\mu_n - \frac{(1 - \nu_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn} + \dfrac{\mathrm{Lu}_p}{\mathrm{Lu}}\sigma_j}{\mathrm{Bi}_m}\,\nu_j\,\mu_n \sin \nu_j\,\mu_n,$$

$$Q_{nj} = \left[1 - \frac{1}{\mathrm{Bi}_q} + (1 - \nu_j^2)\,K_1\right]\sin \nu_j\,\mu_n + \frac{\nu_j\,\mu_n}{\mathrm{Bi}_q}\cos \nu_j\mu_n,$$

$$P_{nj} = -\frac{(1 - \nu_j^2) + \varepsilon\,\mathrm{Ko}\,\mathrm{Pn} + \dfrac{\mathrm{Lu}_p}{\mathrm{Lu}}\sigma_j}{\mathrm{Bi}_m}(\nu_j\,\mu_n \cos \nu_j\,\mu_n - \sin \nu_j\mu_n) +$$

$$+ (1 - \nu_j^2)\sin \nu_j\,\mu_n,$$

$$\chi_{nj} = \sigma_j \sin \nu_j\,\mu_n.$$

The average values of dimensionless transfer potentials for a sphere are

$$\bar{\varphi}(\mathrm{Fo}) = 3\int_0^1 X^2\,\varphi(X, \mathrm{Fo})\,dX.$$

After the necessary operations we get

$$\bar{T} = 1 + \sum_{n=1}^{\infty}\sum_{j=1}^{3} D_{nj}\exp(-\mu_n^2\mathrm{Fo}), \tag{1.62}$$

$$\bar{\theta} = 1 - \frac{1}{\varepsilon\,\mathrm{Ko}}\sum_{n=1}^{\infty}\sum_{j=1}^{3} D_{nj}(1 - \nu_j^2)\exp(-\mu_n^2\mathrm{Fo}), \tag{1.63}$$

$$\bar{P} = \frac{1}{\varepsilon\,\mathrm{Bu}}\sum_{n=1}^{\infty}\sum_{j=1}^{3} D_{nj}\,\sigma_j \exp(-\mu_n^2\,\mathrm{Fo}), \tag{1.64}$$

where

$$D_{nj} = C_{nj}\frac{3(\sin \nu_j\,\mu_n - \nu_j\,\mu_n \cos \nu_j\,\mu_n)}{\nu_j^2\,\mu_n^2}.$$

For practical calculations, equations (1.59) to (1.61) and (1.63) to (1.64) can be simplified using only one or two terms of the series.

2. NON-STATIONARY FIELDS OF ENERGY AND MASS-TRANSFER POTENTIALS WITH BOUNDARY CONDITIONS OF THE SECOND KIND

In a number of high intensity processes the mass flux of vaporized material, $q_m(\tau)$, changes continuously with time. In this case the boundary conditions for one-dimensional problems should be written in the following form:

$$-a_q c_q \gamma_0 \frac{\partial t(R,\tau)}{\partial x} + \alpha\left[t_c - t(R,\tau)\right] - (1-\varepsilon)\rho q_m(\tau) = 0,$$

$$a_m c_p \gamma_0 \frac{\partial \theta(R,\tau)}{\partial x} + a_m c_m \gamma_0 \delta \frac{\partial t(R,\tau)}{\partial x} +$$

$$+ a_m c_m \gamma_0 \frac{\partial p(R,\tau)}{\partial x} + q_m(\tau) = 0$$

$$p(R,\tau) = \varphi(\tau).$$

In order to generalize the problem, transfer of mass in the liquid phase is here separated from transfer of mass in the vapor phase. The same distinction will be made in the system of differential equations of high intensity heat and mass transfer. Moreover, in order to simplify the notation, transfer potentials will be denoted by θ_i, thus $t = \theta_1$, $\theta = \theta_2$ and $p = \theta_3$. The entire system becomes

$$\frac{\partial \theta_1}{\partial \tau} = a_1 \frac{\partial^2 \theta_1}{\partial x^2} + \frac{\varepsilon \rho c_2}{c_1} \frac{\partial \theta_2}{\partial \tau}, \tag{2.1}$$

$$\frac{\partial \theta_2}{\partial \tau} = a_2\left(\delta \frac{\partial^2 \theta_1}{\partial x^2} + \frac{\partial^2 \theta_2}{\partial x^2}\right) + a_3' \frac{\partial^2 \theta_3}{\partial x^2} + \varepsilon \frac{\partial \theta_2}{\partial \tau}, \tag{2.2}$$

$$\frac{\partial \theta_3}{\partial \tau} = a_3 \frac{\partial^2 \theta_3}{\partial x^2} + \varepsilon \gamma_0 \frac{c_2}{c_3} \frac{\partial \theta_3}{\partial \tau}, \tag{2.3}$$

$$\theta_i(x, 0) = f_i(x),$$

$$(0 < x < R) \quad i = 1, 2, 3, \tag{2.4}$$

$$\frac{\partial \theta_i(0,\tau)}{\partial x} = 0, \tag{2.5}$$

$$-a_1 c_1 \gamma_0 \frac{\partial \theta_1(R,\tau)}{\partial x} + \alpha\left[\theta_{1c} - \theta_1(R,\tau)\right] -$$

$$-(1-\varepsilon)\rho\, q_m(\tau) = 0, \tag{2.6}$$

$$a_2 c_3 \gamma_0 \frac{\partial \theta_2(R,\tau)}{\partial x} + a_2 c_2 \gamma_0 \delta \frac{\partial \theta_1(R,\tau)}{\partial x} +$$

$$+ a_2' c_2 \gamma_0 \frac{\partial \theta_3(R,\tau)}{\partial x} + q_m(\tau) = 0, \tag{2.7}$$

$$\theta_3(R,\tau) = \varphi_3(\tau), \tag{2.8}$$

where subscripts, 1, 2, 3 refer to q, m and p, respectively. a' denotes mass transfer in the liquid phase. The solution of this problem was given by A. P. Prudnikov. We cite only the general method of solving this problem.

First let us consider the auxiliary problem, with the functions $\theta_1(R,\tau) = \varphi_1(\tau)$, $\theta_2(R,\tau) = \varphi_2(\tau)$, unknown, i.e.,

$$\theta_k(R,\tau) = \varphi_k(\tau) \quad (k = 1, 2), \tag{2.9}$$

are data which will be determined somewhat later. The solution of the auxiliary problem is now reduced to the determination of functions $\theta_i(x,\tau)$, satisfying the system of equations (2.1) to (2.3), initial conditions (2.4), conditions of symmetry (2.5) and boundary conditions (2.8) and (2.9). The solution is found by simultaneous application of integral Fourier and Laplace transformations. We use the finite Fourier cosine transformation with respect to the variable, x.

$$\{\theta_i(p,\tau)\}_F = \int_0^R \theta_i(x,\tau) \cos\left(n + \frac{1}{2}\right)\frac{\pi}{R} x\, dx. \tag{2.10}$$

The function $\theta_i(x,\tau)$ corresponding to a given transformation $\{\theta_i(p,\tau)\}_F$ is determined by means of the following series:

$$\theta_i(x,\tau) = \frac{2}{R}\sum_{n=0}^{\infty} \{\theta_i(p,\tau)\}_F \cos\left(n + \frac{1}{2}\right)\frac{\pi}{R} x.$$

For the transformation of the second derivative the following formula is used:

$$\left\{\frac{\partial^2 \theta_i(p,\tau)}{\partial x^2}\right\}_F = -\left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{R^2}\{\theta_i(p,\tau)\}_F +$$

$$+ (-1)^n \left(n + \frac{1}{2}\right)\frac{\pi}{R}\, \theta_i(R,\tau) - \frac{\partial \theta_i(0,\tau)}{\partial x}. \tag{2.11}$$

We also apply the Laplace transformation with respect to the variable x:

$$\{\theta_i(x,s)\}_L = \int_0^\infty \theta_i(x,\tau)\, e^{-s\tau}\, d\tau, \tag{2.12}$$

then,

$$\left\{\frac{\partial \theta_i(x,s)}{\partial \tau}\right\}_L = s\{\theta_i(x,s)\}_L - \theta_i(x,0). \tag{2.13}$$

By applying the integral transformations (2.10) and (2.12) and considering equations (2.11) and (2.13), initial conditions (2.14) and boundary conditions (2.18) and (2.19), the original problem is reduced to an algebraic system, whence by means of the appropriate inverse integral transformations we find

$$\theta_i(x,\tau) = \sum_{j=1} (A^i_{js} v_{js} + B^i_{js} w_{js}) \quad (i,s = 1, 2, 3). \tag{2.14}$$

Here, A^i_{js}, and B^i_{js} are constants which depend on the parameters appearing in system of equations (2.1) to (2.3) and which are analogous to the corresponding coefficients given in Section 7 of Chapter IV.

$$v_{js} = \frac{2}{R}\sum_{n=0}^{\infty} \exp\left[-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{R^2} \nu_j \tau\right] \times$$

$$\times \int_0^R f_s(\xi) \cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}\xi\, d\xi \cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}x, \tag{2.15}$$

$$w_{js} = \frac{2}{R}\sum_{n=0}^{\infty} \exp\left[-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{R^2} \nu_j \tau\right] \overset{\tau}{*} (-1)^n \times$$

$$\times \left(n+\frac{1}{2}\right)\frac{\pi}{R}\varphi_s(\tau) \cos\left(n+\frac{1}{2}\right)\frac{\pi}{R}x, \tag{2.16}$$

where $\nu_j e^2$ are roots of the cubic equation

$$Ap^3 + Bp^2 + Cp + D = 0,$$

$$A = 1 - \varepsilon,$$

$$B = e^2 \left[(1-\varepsilon)(a_1+a_3) + \frac{a_2 \delta \varepsilon \rho c_1}{c_2} + \frac{a_2' \varepsilon \gamma_0 c_2}{c_3} + a_2 \right],$$

$$C = e^4 \left[a_1 a_3 (1-\varepsilon) + a_2 a_3 + a_1 a_2 + \frac{a_2 \delta \varepsilon \rho c_2 a_3}{c_1} + \right.$$

$$\left. + a_1 a_2 \varepsilon \gamma_0 \frac{c_2}{c_3} \right],$$

$$D = e^6 a_1 a_2 a_3,$$

$$e = \left(n + \frac{1}{2} \right) \frac{\pi}{R}.$$

It has been assumed here that coefficients of the system of differential equations satisfy the following condition:

$$\begin{vmatrix} p + a_1 e^2 & -\frac{\varepsilon \rho c_2}{c_1} p & 0 \\ a_2 \delta e^2 & (1-\varepsilon) p + a_2 & a_2' e^2 \\ 0 & \frac{\varepsilon \gamma_0 c_2}{c_3} p & p + a_3 e^2 \end{vmatrix} \neq 0.$$

The symbol $\overset{\tau}{*}$ denotes the convolution operation

$$f(\tau) \overset{\tau}{*} g(\tau) = \int_0^\tau f(\xi)\, g(\tau-\xi)\, d\xi = \int_0^\tau f(\tau-\xi)\, g(\xi)\, d\xi = g(\tau) \overset{\tau}{*} f(\tau).$$

In order to simplify the following, we introduce the theta function, in terms of which equations (2.15) and (2.16) become the following:

$$v_{js} = \frac{1}{2R} \int_0^R \left\{ \vartheta_2 \left[\frac{x-\xi}{2R}, \frac{\nu_j \tau}{R^2} \right] + \vartheta_2 \left[\frac{x+\xi}{2R}, \frac{\nu_j \tau}{R^2} \right] \right\} f_s(\xi)\, d\xi,$$

$$w_{js} = \frac{1}{R} \frac{\partial}{\partial x} \int_0^\tau \vartheta_2 \left[\frac{R-x}{2R}, \frac{\nu_j (\tau-\eta)}{R^2} \right] \varphi_s(\eta)\, d\eta, \qquad (2.17)$$

$$\vartheta_2(x, \tau) = \frac{1}{\sqrt{\pi \tau}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[-\frac{(x+n)^2}{\tau}\right]. \tag{2.18}$$

The unknown functions $\varphi_k(\tau)$ $(k = 1, 2)$ can now be determined from boundary conditions (2) to (6) and (2) to (7). Differentiating equation (2.14) with respect to x, and replacing the second derivative of the θ-function, ϑ_2, with respect to x by its derivative with respect to τ, we get

$$\frac{\partial \theta_i(x, \tau)}{\partial x} = \sum_{j=1}^{3} (A^i_{js} P_{js} + B^i_{js} Q_{js}) \quad (i, s = 1, 2, 3), \tag{2.19}$$

where

$$P_{js}(x, \tau) = \frac{1}{2R} \int_0^R \frac{\partial}{\partial x} \left\{ \vartheta_2 \left[\frac{x-\xi}{2R}, \frac{\nu_j \tau}{R^2}\right] + \right.$$

$$\left. + \vartheta_2 \left[\frac{x+\xi}{2R}, \frac{\nu_j \tau}{R^2}\right] \right\} f_s(\xi)\, d\xi,$$

$$Q_{js}(x, \tau) = \frac{1}{\nu_j R} \int_0^\tau \frac{\partial}{\partial(\tau - \eta)} \times$$

$$\times \vartheta_2 \left[\frac{R-x}{2R}, \frac{\nu_j(\tau-\eta)}{R^2}\right] \varphi_s(\eta)\, d\eta. \tag{2.20}$$

From equation (2.18) it is easy to show that the derivative $\frac{\partial}{\partial(\tau-\eta)} \vartheta_2$, which exists for all $x \neq R$, tends to infinity, as $\tau^{-\frac{3}{2}}$ as $\tau \to 0$, when $x = R$ and $\tau \to \eta$. In order to avoid any difficulties, we replace conditions (2.6) and (2.7) by other conditions which are obtained by integrating them with respect to τ. Instead of equations (2.6) and (2.7) we then get

$$-a_1 c_1 \gamma_0 \int_0^\tau \frac{\partial \theta_1(R, \eta)}{\partial x} \partial \eta + \alpha \int_0^\tau [\theta_{1c} - \theta_1(R, \eta)]\, d\eta -$$

$$-(1 - \varepsilon) \rho \int_0^\tau q_m(\eta)\, d\eta = 0. \tag{2.21}$$

$$a_2 c_2 \gamma_0 \left[\delta \int_0^\tau \frac{\partial \theta_1(R,\eta)}{\partial x}\, d\eta + \int_0^\tau \frac{\partial \theta_2(R,\eta)}{\partial x}\, d\eta \right] +$$

$$+ a_2' c_3 \gamma_0 \int_0^\tau \frac{\partial \theta_3(R,\eta)}{\partial x}\, d\eta + \int_0^\tau q_m(\eta)\, d\eta = 0. \qquad (2.22)$$

Integrating (2.19) with respect to τ, we find

$$\int_0^\tau \frac{\partial \theta_i(x,\eta)}{\partial x}\, d\eta = \sum_{j=1}^{3} (A^i_{js} P^*_{js} + B^i_{js} Q^*_{js})\ (i, s=1,2,3), \qquad (2.23)$$

where

$$P^*_{js}(x,\tau) = \int_0^\tau P_{js}(x,\eta)\, d\eta, \quad Q^*_{js}(x,\tau) = \int_0^\tau Q_{js}(x,\eta)\, d\eta.$$

Integrating equation (2.20) with respect to τ and changing the order of integration by means of the Dirichlet formula, we get

$$Q^*_{js}(x,T) = \frac{1}{\nu_j R} \int_0^T d\tau \int_0^\tau \varphi_s(\eta) \frac{\partial}{\partial(\tau-\eta)} \times$$

$$\times \vartheta_2 \left[\frac{R-x}{2R}, \frac{\nu_j(\tau-\eta)}{R^2} \right] d\eta = \frac{1}{\nu_j R} \times$$

$$\times \int_0^T \varphi_s(\eta)\, d\eta \int_\eta^T \frac{\partial}{\partial(\tau-\eta)} \vartheta_2 \left[\frac{R-x}{2R}, \frac{\nu_j(\tau-\eta)}{R^2} \right] d\tau =$$

$$= \frac{1}{\nu_j R} \int_0^T \varphi_s(\eta)\, d\eta \left\{ \vartheta_2 \left[\frac{R-x}{2R}, \frac{\nu_j(\tau-\eta)}{R^2} \right] - \vartheta_2 \left[\frac{R-x}{2R}, 0 \right] \right\}.$$

From the above it follows that at all $x < R$

$$Q^*_{js}(x,T) = \frac{1}{\nu_j R} \int_0^T \varphi_s(\eta) \times$$

$$\times \vartheta_2 \left[\frac{R-x}{2R}, \frac{\nu_j(T-\eta)}{R^2} \right] d\eta. \qquad (2.24)$$

By means of direct calculations it can be shown that equation (2.24) applies also at $x = R$, i.e.,

$$Q_{js}^{*}(R, T) = \frac{1}{\nu_j R} \int_0^T \varphi_s(\eta)\, \vartheta_2 \left[0, \frac{\nu_j (T - \eta)}{R^2} \right] d\eta. \qquad (2.25)$$

This integral converges, because according to equation (2.18) the function ϑ_2 tends to infinity, as $T^{-1/2}$ when $\eta \to T$ and $T \to 0$. Forming, by means of equations (2.23) to (2.25), the corresponding terms in equations (2.21) and (2.22) and considering equation (2.9) we get a system of integral Volterra equations of the first kind:

$$\sum_{\beta=1}^{2} \int_0^{\tau} K_{\alpha\beta}(\tau, \eta)\, \varphi_\beta(\eta)\, d\eta = g_\alpha(\tau) \quad (\alpha = 1, 2), \qquad (2.26)$$

where the functions $K_{\alpha\beta}(\tau, \eta)$ and $g_\alpha(\tau)$ are of such a form that the system of equations (2.26) can be reduced to a system of generalized Abel equations, and the latter to the system of integral Volterra equations of the second kind as shown in Section 7, of Chapter IV:

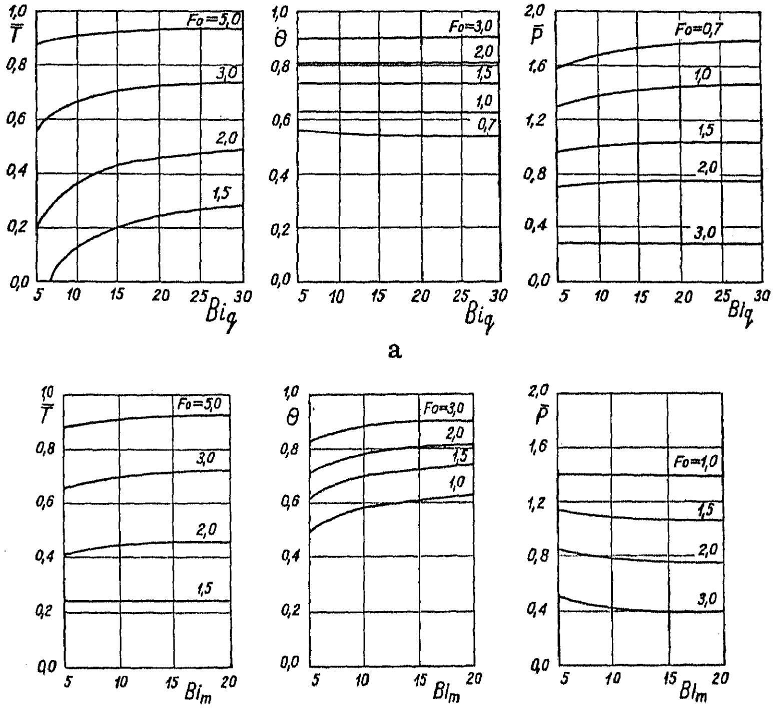
$$\varphi_\alpha(\tau) + \sum_{\beta=1}^{2} \chi_{\alpha\beta}(\tau, \eta)\, \varphi_\beta(\eta)\, d\eta = \omega_\alpha(\tau) \quad (\alpha = 1, 2). \qquad (2.27)$$

The solution of this problem is obtained by finding $\varphi_1(\tau)$ and $\varphi_2(\tau)$ from equation (2.27), and substituting them in equalities (2.24) and (2.17).

3. THE EFFECT OF DIMENSIONLESS NUMBERS ON HIGH INTENSITY HEAT AND MASS TRANSFER

The solutions given in Sections 1 and 2 show that the distributions of dimensionless energy and mass-transfer potentials, as well as their rates change, are functions of a large group of dimensionless numbers. In addition to the dimensionless numbers noted in Section 6 of Chapter II, new dimensionless numbers affect high intensity heat and mass transfer. These new dimensionless numbers (Bu, Lu_p) characterize macroscopic transfer. The change in the transfer mechanism, related to the occurrence of macroscopic mass and heat flow, is also reflected in the previously described group of dimensionless numbers. Thus, the Pn and Ko numbers, which describe internal heat and mass transfer, no longer completely characterize the distributions of some transfer potentials.

However, the dimensionless numbers Bi_q and Bi_m which describe molecular heat and mass transfer on the surface, play the same role when there is macroscopic heat and mass transfer, as when there is molecular transfer alone. Low values of the Biot numbers correspond to incompletely developed potential fields and low rates of transfer of mass and energy. At values of the Biot number around 1.0, large temperature and mass gradients appear in the material.



a

b

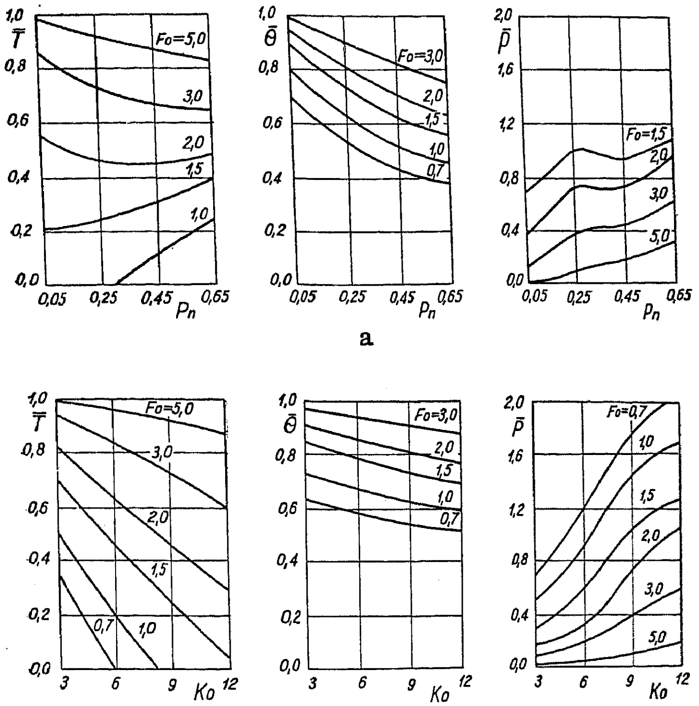
Fig. 53. The dependence of transfer potentials a — on Bi_q; b — on Bi_m.

As a result, the redistribution of heat and mass in the body is intensified. At the same time, considerable stresses are developed, causing the material to warp and crack and promoting the formation of pores. As with molecular transfer, after a quasi-stationary regime is reached, i.e., beginning with $Fo \approx 0.7 - 1.0$, the heat transfer Biot number begins to affect only the thermal field (Fig. 53a) while the mass transfer Biot number affects only the dimensionless mass field (Fig. 53b).

The filtration potential pressure distribution becomes independent of both Biot numbers. The lack of influence of the Biot numbers on the filtration potential, and consequently on the macroscopic transfer, indicates that the dimensionless numbers describing surface heat and mass transfer are basically related to the molecular transfer mechanism. Examination of Figure 53 shows that the dimensionless mass concentration field is a weaker function of the Bi_m number than the dimensionless temperature field is of the Bi_q number; this reflects the greater "inertia" of the mass transfer potential in comparison with the heat transfer potential at $\mathrm{Lu}<1$. Analysis of the solutions obtained also shows that during the stationary regime the rates of change of the mass transfer potential and of the filtration potential pressure decrease linearly with increasing $\bar{\Theta}$, and are not functions of Bi_q.

In a three-dimensional coordinate space $(\bar{T}, \bar{\Theta}, \bar{P})$ high-intensity heat and mass transfer processes are described by an open (non-intersecting) curve, whose every point corresponds to a certain generalized time (Fo). By changing one of the dimensionless numbers, we shift the process curve, and as it shifts it sweeps out a process surface which describes the dimensionless number under consideration. The lines of intersection of process surfaces whith coordinate surfaces represent the dependence of single potentials on a given dimensionless number when the other potentials are held constant. The uniqueness of the dependence so deduced permits us to draw certain conclusions about the influence of transfer potentials on the dimensionless numbers; on this basis the experimental and theoretical data can be compared. The change in the dimensionless numbers describing surface heat and mass transfer with increasing $\bar{\Theta}$ is in good agreement with experimental results of P. D. Lebedev and A. F. Chizhskiy.

The presence of macroscopic transfer in addition to molecular mass transfer changes the dependence of the distributions of dimensionless potentials on the dimensionless numbers of internal heat and mass transfer (Pn, Ko). Section 6 of Chapter II, which did not consider macroscopic transfer, showed the independence of Pn on the temperature field. Macroscopic transfer results in the appearance of a new relationship between the temperature distribution and the Pn number. At low values of Fo the temperature of the porous medium increases considerably with increasing Pn; however, this relationship changes with time and at values of the Fourier number greater than 2 it is reversed (Fig. 54a). The dependence of the filtration potential on the Pn number, together with the new relationship between this dimensionless number and the temperature distribution, shows that the Posnov number is related to both the molecular and the macroscopic transfer mechanisms. An analogous conclusion can be drawn with respect to the Kossovich number (Fig. 54b). The occurrence of strong macroscopic transfer produces in the first place a redistribution of mass; redistribution of heat, however, is a result of the redistribution of mass. The Posnov number describing



a

b

Fig. 54. Dependence of transfer potentials on a — Pn; b — Ko.

the internal mass transport by molecular transfer should reflect the occurrence of the new mechanism much more strongly than the Kossovich number. The latter describes internal heat transfer processes. With increasing Pn the dimensionless transfer potential decreases. This shows, that during equal intervals of time a larger amount of mass is removed from the material at lower values of the Posnov number. The influence of the Ko number appears first in the temperature field; in this respect the previously noted relationship and similarity between the Kossovich number and the Biot number is preserved. The lowering of the average temperature of the material with increasing Ko number, and also the insignificant effect of Ko upon the drying rate, shows that drying processes should be designed and operated at large Ko number to prevent decomposition of the material.

A new dimensionless number for high intensity heat and mass transfer is the Bulygin number, $\mathrm{Bu}=\frac{\rho c_p}{c_q}\frac{p_0}{t_c-t_0}$. This number is one of the group of dimensionless numbers describing internal heat and mass transfer. The Bu number, like the Kossovich number, describes the accumulation capacity of the body, but in contrast to the latter it specifies not the entire amount of heat expended for vaporization, but only the amount of heat expended in producing that fraction of vapor which takes part in the macroscopic transfer. The Bu number refers only to the macroscopic transfer and therefore it influences only the distribution of the pressure potential. It influences neither the temperature nor the mass distribution. The dependence of the pressure potential on the Bu number is shown in Figure 55. The Bu number varies over a wide range. The value of the average pressure potential is inversely proportional to the Bulygin number.

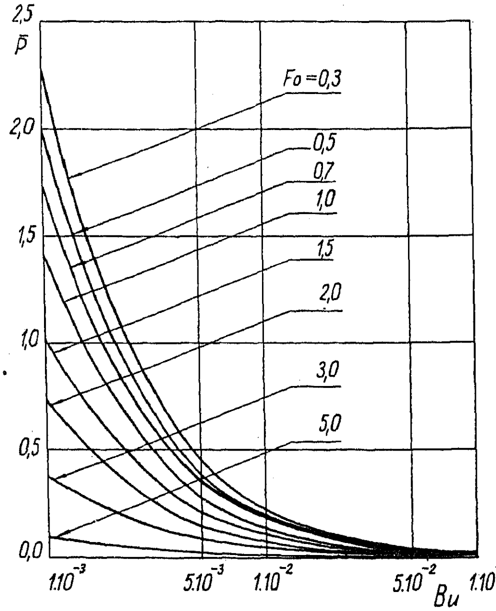


Fig. 55. **Filtration potential pressure as a function of Bu.**

Another dimensionless quantity appearing when there is intensive macroscopic heat and mass transfer is the Lu_p number, which characterizes the ratio between macroscopic mass transfer and heat transfer, i.e., the inertia of the filtration potential field compared to that of the temperature field. In this respect the Lu_p is analogous to the Lu number. Figure 56 shows that the Lu_p number is independent of T and Θ. This number ranges from several hundredths

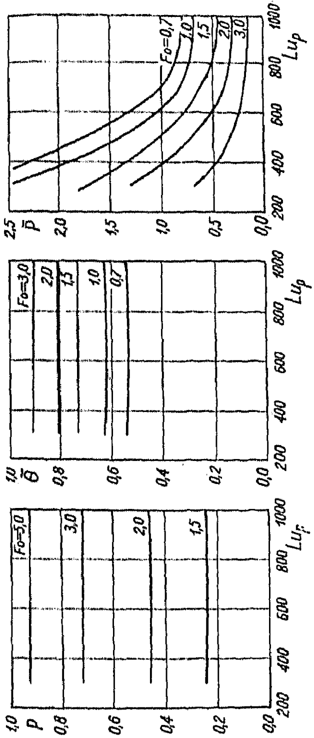


Fig. 56. Transfer potentials as a function of Lu$_p$.

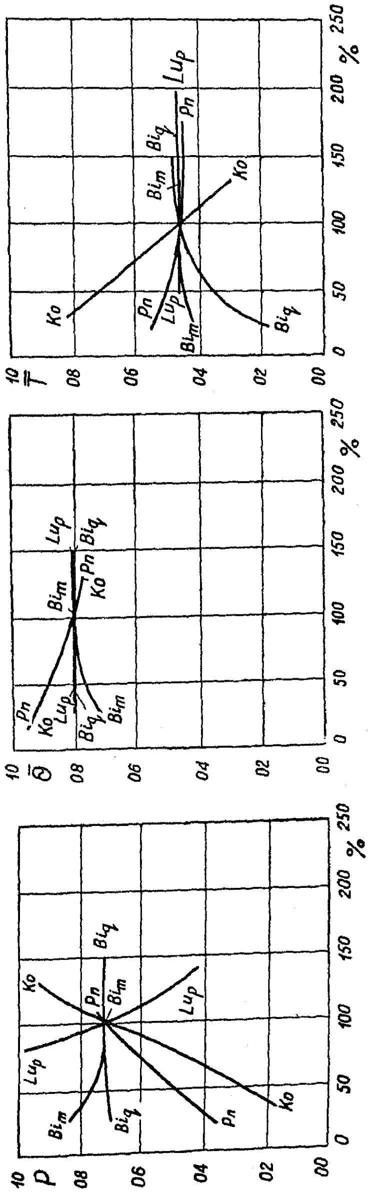


Fig. 57. Specific effect of dimensionless numbers on high intensity heat and mass transfer.

in convective heating to several tens of thousands in high frequency heating of moist materials. In connection with the analysis of the dependence of heat and mass transfer on the dimensionless numbers of macroscopic transport we shall dwell in somewhat more detail on the dynamics of change of the filtration potential in general, and on the non-relaxing pressure excess in particular.

During intense heating of a body of moist porous material there is rapid vaporization which leads to the formation of a persistent pressure gradient. This gradient depends on the relative magnitudes of the time needed for relaxation of the excess pressure through the skeleton of the material, and the time required for vaporization to re-establish the original state. Depending on the method of heating the body and also on the values of thermal, physical, and mechanical properties of the material, the distribution of the filtration potential can assume several forms. In any case, however, there is strong increase of the filtration potential during the initial stage of the process. With decreasing mass concentration, and with increasing $\bar{\Theta}$, the intensity of the internal vapor formation decreases and this leads to a decrease of the filtration potential gradient. The magnitude of the filtration potential and the way in which it changes depend differently on different dimensionless numbers. As a first approximation, the specific influence of each of the dimensionless numbers can be evaluated from the magnitude of the changes they produce. With the reference value of 100 for each of the dimensionless numbers, Figure 57 and calculations (not shown on the figure) indicate that the filtration potential is primarily a function of the following dimensionless numbers: Lu_p, Lu and Bu. The deciding role is played here by the Lu number. Thus, for example, if $\bar{P}= 0.7268$ at $\mathrm{Lu}= 0.3$, then it corresponds that $\bar{P}= 60.57$ at $\mathrm{Lu}= 0.7$, and $\bar{P} = 83.95$ at $\mathrm{Lu}= 1.0$. The remarkable peculiarity of the filtration potential is that it reacts much more strongly to changes in the magnitude of dimensionless numbers than do the heat and mass transfer potentials. This circumstance constitutes the basic difficulty in controlling the value of $\bar{P}$ in actual processes.

On the basis of this analysis and also by considering Figure 57 we can propose simplified dimensionless equations for the description of internal heat and mass transfer of high intensity. Such equations are

$$\begin{aligned} \bar{T}&=\bar{T}(\mathrm{Lu}, \mathrm{Bi}_q, \mathrm{Ko}, \mathrm{Pn}, \varepsilon, \mathrm{Fo}),\\ \bar{\Theta}&=\bar{\Theta}\,(\mathrm{Lu}, \mathrm{Bi}_m, \mathrm{Ko}, \mathrm{Pn}, \varepsilon, \mathrm{Fo}),\\ \bar{P}&=\bar{P}\,(\mathrm{Lu}, \mathrm{Lu}_p, \mathrm{Bu}, \mathrm{Ko}, \mathrm{Pn}, \varepsilon, \mathrm{Fo}). \end{aligned} \tag{3.1}$$

Hence experimental data on the kinetics of high-intensity heating of porous materials should be presented in the dimensionless forms suggested by equation (3.1). Depending on the circumstances of a given process, certain of the dimensionless numbers may be controlling, and the rest of them can be neglected.

Nomenclature

$f_L(\xi, s) = \int_0^\infty f(\xi, \tau)\, e^{-s\tau}\, d\tau$ — Laplace transformation

t — temperature, degrees

θ — mass transfer potential, potential units

τ — time, hours

ξ — generalized coordinate, meters

R — value of the coordinate on the body surface

a_q — thermal diffusivity, m^2/hr

a_m — coefficient of potential conductivity of mass transfer, m^2/hr

δ — thermal gradient coefficient, or Soret coefficient, degrees xx

c_q — specific heat capacity of moist body, Kcal/kg-°C

C — heat capacity of the external medium (heat carrier) Kcal/m^3-°C

γ — density of the dry portion of the moist body, kg/m^3

v — velocity of heat carrier in the layer, m/hr

P — porosity of the layer, or relative pressure

α_q — heat transfer coefficient, Kcal/m^2 hr -°C

α_m — mass transfer coefficient km/m^2-hr-potential unit

β — mass transfer coefficient, related to the difference of mass concentration, m/hr

ρ — specific heat of phase transformation, Kcal/kg

Γ — shape factor

$T = \frac{t(\xi, \tau) - t(\xi, 0)}{t_c - t(\xi, 0)}$ — dimensionless temperature

$\theta = \frac{\theta(\xi, \tau) - \theta(\xi, 0)}{\theta(\xi, 0) - \theta_p}$ — dimensionless mass transfer potential

$\mathrm{Fo} = \frac{a_q \tau}{R^2}$ — Fourier number

$\mathrm{Lu} = \frac{a_m}{a_q}$ — Lukomskiy number (Loo-KUHM-skee)

$\mathrm{Bi}_q = \frac{\alpha_q}{\lambda_q} R, \quad \mathrm{Bi}_m = \frac{\alpha_m}{\lambda_m} R = \frac{\beta R}{a_m}$ — heat and mass transfer numbers, respectively

Pn = Posnov number (Paw-SNAWV) $\frac{\delta\,[t_c - t(0)]}{u(0) - u_p} = \frac{\delta_\theta\,[t_c - t(0)]}{[\theta(0) - \theta_p]}$

Ko = Kossovich number (Kuh-SAW-veech) $\frac{\rho\,[u(0) - u_p]}{c_q\,[t_c - t(0)]} = \frac{\rho\, c_m'\,[\theta(0) - \theta_p]}{c_q\,[t_c - t(0)]}$

ε — phase transformation number

$\mathrm{Fe} = \varepsilon\,\mathrm{Ko}\,\mathrm{Pn} = \frac{\varepsilon \rho \delta}{c_q}$ — Fedorov number (FYUH-duh-rawv)

$\mathrm{Mr} = \frac{a_q}{v c_p \gamma_p}$ — Margulis number (Mahr-GOO-lees)

$\mathrm{Mn} = \frac{F R}{P}$ — Minyovich number (Meen-YAW-veech)

$\mathrm{Pe} = \frac{v R}{a_q}$ — Peclet number

$\mathrm{Pd} = \frac{b R^2}{a_q [t_c - t(\xi, 0)]}$, $\mathrm{Pd}' = \frac{\varkappa R^2}{a_q}$ —Predvoditelev number (Prehd-voh-DEE-tyeh-lehv)

$W = \frac{t_n - t_\text{ц}}{t_c - t_n}$; $V = \frac{\theta_\text{ц} - \theta_n}{\theta_n - \theta_p}$ — Dimensionless parameters of non-uniformity of initial temperature and mass-transfer potential distributions

$W_c = \frac{t_c}{t_c - t_0}$, $W_0 = \frac{t_0}{t_c - t_0}$ — dimensionless temperature parameters

$K_1 = \frac{1 - \varepsilon}{\varepsilon} \mathrm{Lu} \frac{\mathrm{Bi}_q}{\mathrm{Bi}_m}$, $K_2 = V - \mathrm{Pn}\ \theta$ — complex dimensionless numbers

$\mathrm{Bu} = \frac{\rho c_p p_o}{c_q (t_c - t_0)}$ — Bulygin number (Boo-LIH-gheen)

SUBSCRIPTS

L — Laplace transformation symbol
F — Fourier transformation symbol
o — initial conditions
c — surrounding medium
п — surface
ц — center of the body
p — equilibrium value
q — heat transfer
m — mass transfer
j — subscript assuming two values: $j = 1; 2$.

TRIGONOMETRIC EQUIVALENTS

tg = tan
ctg = cot
sh = sinh
ch = cosh

Bibliography

1. Grew, K.E. and T.L. Ibbs: Termicheskaya diffuziya v gazakh (Thermal Diffusion in Gases), Gostekhizdat, 1956 (Transl. from English).
2. Grinberg, G.A.: Izbrannyye voprosy matematicheskoy teorii elektricheskikh i magnitnykh yavleniy (Selected Mathematical Problems in the Theory of Electrical and Magnetic Phenomena), Ac. of Sci. of USSR, 1948.
3. De Groot, S.R.: Termodinamika neobratimykh protsessof (Thermodynamics of Irreversible Processes). Gostekhizdat, 1956 (Transl. from English).
4. Denbigh, K.: Termodinamika statsionarnykh neobratimykh protsessov (Thermodynamics of Steady-State Irreversible Processes), Foreign Literature Press, 1954.
5. Ditkin, V.A. and A.P. Prudnikov: Operatsionnoye ischisleniye po dvum peremennym i yego prilozheniya (Operational Calculations of Two Variables and its Application), Fizmatizdat, 1958.
6. Zhuchkov, P.A.: Sushka torfa v sloye, dvizhushchemsya v potoke gaza (Drying of a Peat Layer Moving in a Gas Stream), Kotloturbostroyeniye (Boiler and Turbine Design), No. 5, 1951.
7. Kichkina, E.S.: Protsessy sushki i teploobmena v sloye kuskovykh materialov (Drying and Heat Transfer in Layers of Fragmental Materials), Dissertation, Leningrad Polytechnic Institute, 1947.
8. Koshlyakov, N.S.: Osnovnyye differentsial'nyye uravneniya matematicheskoy fiziki (Basic Differential Equations in Mathematical Physics), ONTI, 1936.
9. Krylov, A.N.: O nekotorykh differentsial'nykh uravneniyakh matematicheskoy fiziki, imeyushchikh primeneniye v tekhnicheskikh voprosakh (Application of Certain Differential Equations of Mathematical Physics to Engineering Problems), Ac. of Sci. of USSR, 1938.
10. Kuznetsov, S.P.: Nagrevaniye i okhlazhdeniye vlazhnykh tel (Heating and Cooling of Moist Bodies), Dissertation, 1952.
11. Lebedev, P.D.: Sushka infrakrasnymi luchami (Drying by Infrared Rays), Gosenergoizdat, 1955.
12. Levin, V.I., Yu.I. Grosberg: Differentsial'nyye uravneniya matematicheskoy fiziki (Differential Equations in Mathematical Physics), Gostekhizdat, 1951.

13. Lykov, A.V.: Novyy metod opredeleniya koeffitsientov temperaturoprovodnosti vlazhnykh materialov (New Method of Determining Thermal Conductivity of Moist Materials), Zh.E.T.E. (Journal of Experimental and Engineering Physics), No. 2, 1935.

14. ________: Osnovnyye zakonomernosti teplo- i massobmena v protsesse sushki (Basic Law of Heat and Mass Transfer in Drying), Izv. V.T.I. (Bulletin of the All-Union Power Engineering Institute), No. 10, 1952.

15. ________: Teoriya teploprovodnosti (Theory of Heat Conduction), Gostekhizdat 1952.

16. ________: Yavleniya perenosa v kapillyarno-poristykh telakh (Transfer Phenomena in Capillary-Porous Bodies), Gostekhizdat, 1954.

17. Lykov, A.V., M.S. Smirnov: K issledovaniyu kinetiki protsessa sushki (Kinetics of Drying Processes), Izvestiya A.N. SSSR (Bulletin of the Academy of Science of the USSR, Engineering Science Series) No. 8, 1956.

18. Lykov, A.V.: Teplo- i massoobmen v protsessakh sushki (Heat and Mass Transfer in Drying), Gosenergoizdat, 1956.

19. Lykov, A.V., A.P. Prudnikov: K issledovaniyu yavleniy perenosa tepla i veshchestva v poristykh sredakh (Heat and Mass Transfer Phenomena in Porous Media), D.A.N., B.S.S.R. (Proceedings of the Academy of Science of the Byelorussian S.S.R.), 2, 334 (1959).

20. Lykov, A.V.: Teplo- i massoobmen v dispersnykh sredakh pri fazovykh prevrashcheniyakh (Heat and Mass Transfer in Porous Media with Phase Transformation), T.F.Zh. (Journal of Engineering Physics) No. 6, 1958.

21. Mikhaylov, Yu.A.: Primeneniye sistemy differentsial'nykh uravneniy teplo- i massoobmena k protsessu konvektivnoy sushki (Application of a System of Heat and Mass Transfer Differential Equations to Drying by Convection), Izvestiya A.N. Latv. SSR (Bulletin of the Academy of Science of the Latvian S.S.R.) No. 12, 1956.

22. ________: Vliyaniye kriteriyev podobiya na teplo- i massoobmen pri konvektivnoy sushke (The Influence of Dimensionless Numbers of Heat and Mass Transfer in Drying by Convection), Op. cit., No. 6, 1957.

23. ________: K teorii protsessa konvektivnoy sushki (Theory of Drying by Convection), I.F. Zh. (Journal of Engineering Physics), No. 1, 1958.

24. Polonskaya, F.M.: Issledovaniye temperaturnogo polya vlazhnykh materialov v protsesse sushki (period postoyannoy skorosti), (Temperature Distribution in Moist Materials during Drying), (in the Constant Rate Period), Zh.T.F. (Journal of Engineering and Physics), Issue 5, 1952.

25. Prudnikov, A.P.: Analiticheskoye issledovaniye protsessov teplo-i massoobmena (dissertatsiya), (Analytical Investigation of Heat and Mass Transfer), Dissertation, Academy of Science of the Byelorussian SSR, 1957.
26. ________: Resheniye nekotorykh zadach teorii molekulyarnogo perenosa dlya tel prosteyshey geometricheskoy formy (Solutions of Certain Problems in the Theory of Molecular Transfer for Simple Bodies), Izvestiya A.N. SSSR (Bulletin of the Academy of Science of the USSR), No. 8, 1957.
27. ________: K issledovaniyu teplo-i massoobmena v dispersnykh sredakh (Investigation of Heat and Mass Transfer in Porous Media), I.F.Zh. (Journal of Engineering Physics), No. 4, 1958.
28. Shornik: "Teplo- i massoobmen v protsessakh ispareniya," (Symposium on Heat and Mass Transfer during Vaporization), Academy of Science of the USSR, 1958.
29. Sizyakova, E.I.: Issledovaniye kinetiki termovlagoprovodnosti v kapillyarnoporistykh telakh. Teplo- i massoobmen v pishchevykh produktakh (Investigation of the Kinetics of "Thermal Moisture Conductivity" in Capillary-Porous bodies. Heat and Mass Transfer in Food Products), Trudy MTIPP (Transactions of the Moscow Technical Institute of Food Industry), Issue VI, 1956.
30. Smirnov, M.S.: Primeneniye metoda integral'nykh preobrazovaniy k resheniyu zadach teorii molekulyarnogo perenosa (Application of the Integral Transformation Method to the Solution of Problems in Molecular Transfer Theory), Dissertation, 1955.
31. Sneddon, I.: Preobrazovaniya Fourier (Fourier Transformations), Foreign Literature Press, 1955.
32. Temkin, A.G.: Geometricheskiy kriteriy oklazhdeniya (Dimensionless Geometrical Factor in Cooling), Zh.T.F. (Journal of Engineering Physics), No. 4, 1952.
31. ________: Vliyaniye integral'nogo kriteriya formy na protsess teploprovodnosti (Influence of the Integral Shape Number on Heat Conduction), Zh.T.F. (Journal of Engineering Physics), No. 3, 1954.
34. Tikhonov, A.N. and A.A. Samarskiy: Uravneniya matematicheskoy fiziki (Equations of Mathematical Physics), Gostekhizdat, 1953.
35. Tolstov, G.P.: Ryady Fourier (Fourier Series), Gostekhizdat, 1951.
36. Tranter, C.J.: Integral'nyye preobrazovaniya v matematicheskoy fizike (Integral Transforms in Mathematical Physics), Gostekhizdat, 1956. Translated from English.
37. Crank, I.: The Mathematics of Diffusion, Oxford, 1956.
38. Kay, I.M.: An Introduction to Fluid Mechanics and Heat Transfer, Cambridge, 1957.

39. Guggenheim, E.A.: Thermodynamics, Amsterdam, 1957.
40. Eckert, E.R.G., R.M. Drake: Heat and Mass Transfer, New York, 1959.